# Gordon Exercises 2

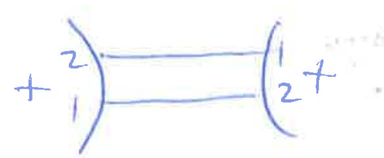
(1) Prove the Parity Rule

(2) Let $S_i$ be an essential 2-sphere in $M(\alpha_i)$, $i=1,2$, with $n_i$ minimal over all such $S_i$.

Show that (i) $F_i$ is incompressible in $M$

(ii) $F_i$ is not parallel into $\partial M$

(iii) no simple closed curve component of $F_1 \cap F_2$ can lie in a disk in $F_1$ or $F_2$.

(3) In the setting of Q(2), suppose $\Gamma_1$ contains a face of the form 

. Show that $M(\alpha_2) \cong \mathbb{R}P^3 \# Q$ for some 3-manifold $Q$.

# GEOMETRIC APPROACHES TO DEHN SURGERY
## Problem sheet

(0) If you have not already done so, download Snappy. Familiarise yourself with it, so that you can form a picture of the cusp neighbourhood of the knots $4_1$ and $6_1$. This gives a view of upper half-space from infinity. The inverse image of the cusp is a collection of horoballs, including $\{(x, y, z) : z \geq 1\}$. A horoball is *full-sized* if its Euclidean diameter is 1. Up to the action of the fundamental group of the cusp, how many full-sized horoballs does $4_1$ have? What about $6_1$?

You will not need Snappy to answer the remaining questions, but it is a very useful tool.

(1) Determine the cusp shape of the exterior of the figure-eight knot. What is the area of the boundary of its maximal cusp?

(2) Show that its shortest slope has length 1, and determine the length of the second shortest slope.

(3) (i) Let $s$ be a Euclidean geodesic on the boundary of a cusp $N = T^2 \times [1, \infty)$. Let $s \times [1, \infty)$ be the vertical annulus over it. Compute the area of $s \times [1, \infty)$ in terms of the length of $s$.

(ii) What is the relationship between the volume of $N$ and the area of $\partial N$?

(4) Let $\mathcal{S}$ be a collection of distinct slopes on the torus.

(i) Show that if $\Delta(s, s') \leq 1$ for each $s, s' \in \mathcal{S}$, then $|\mathcal{S}| \leq 3$.

(ii) Show that if $\Delta(s, s') \leq 2$ for each $s, s' \in \mathcal{S}$, then $|\mathcal{S}| \leq 4$.

(5) Show that for any finite-volume hyperbolic 3-manifold $M$, there is a choice of horoball neighbourhood of $\partial M$ where all the cusps have the same volume. Show that there is a choice of maximal horoball neighbourhood with this property. Prove that this maximal horoball neighbourhood is uniquely defined.

(6) [*Harder*] Let $M$ be the exterior of the figure-eight knot. Show that provided $p/q$ avoids a finite set of slopes, $M(p/q)$ is homeomorphic to $M(p'/q')$ if and only if $p'/q' = \pm p/q$.

# Excercises on Finding Diagrams using SnapPy and KLO

Perspectives on Dehn Surgery, ICERM K. Baker , July 15, 2019

## 1 Example 1

Recall that we have

```
M=Manifold('t12533')
M01=M.drill(0).drill(1)
```

1. Browse `M` and poke around.
2. Check that filling the last two cusps of `M01` along the meridians of the drilled geodesics produces a manifold isometric to `M`.
3. Check that `M` is asymmetric. Is `M01` asymmetric?
4. Find knot diagrams for the following manifolds:

   `'t12681'`, `'o9_38928'`, `'o9_39162'`, `'o9_40363'`,

   `'o9_40487'`, `'o9_40504'`, `'o9_40582'`, `'o9_42675'`

   These census manifolds are complements of asymmetric L-space knots, found by Nathan Dunfield.
5. Which are closures of positive braids? (Up to mirroring.)
6. Use KLO to find the Alexander polynomials.
7. Use SnapPy within Sage to find their Alexander polynomials.

## 2 Example 2

1. Draw the $(2, 6)$ torus link in SnapPy. Find the cabling slope on one component. For each solid torus filling of one component, find the further corresponding $S^3$ fillings of the other component.
2. Generalize DMM's other seiferters of $P(-3, 3, 5)$.
3. In our example, what goes wrong when $n = 2$?
4. Find a link diagram for our example with $n = 3$.
5. From KLO, export a link diagram for use in SnapPy. Load in PLink and send to SnapPy. Is it the manifold you expected?
6. What Seifert fibered spaces are obtained from our examples? Use KLO!

## 3 Example 3

In §4.2 of "A census of exceptional Dehn fillings," Dunfield identifies the following hyperbolic, one-cusped manifolds as having four toroidal fillings.
`'s772'`, `'s778'`, `'s911'`, `'v2640'`, `'t08282'`, `'t11538'`, `'t12033`, `'t12035'`, `'t12036'`, `'t12041'`, `'t12043'`, `'t12045'`, `'t12050'`, `'t12548'`, `'t12648'`, `'o9_35259'`, `'o_936732'`, `'o9_37030'`, `'o9_38039'`, `'o9_39094'`, `'o9_40054'`, `'o9_41000'`, `'o9_41004'`, `'o9_41006'`, `'o9_41007'`, `'o9_41008'`, `'o9_43799'`.

Are there infinitely many such examples?

By visualizing their quotients, we may be able to discern relationships among them that generalize to infinite families. Pick a manifold and try the following:

1. Show the manifold is strongly invertible or freely invertible.
2. Find a tangle diagram for the quotient.
3. Observe how essential tori arise in the double branched covers of the associated fillings.

# Basic Notes on SnapPy

- `M=Manifold()` opens PLink sketcher to input link complement
- `M=Manifold('blahblah')` loads the known manifold "blahblah"
- Press Tab to complete partially typed function or get list of possible completions
- Press ↑ to cycle through previously typed commands. If you've already typed something, it will cycle through commans that complete what you've typed.
- Don't forget () even if there is no argument for the function!
- `M.solution_type()` tells you if SnapPy has found a hyperbolic structure. When if finds one, you've got it. When it doesn't, maybe it was a funny triangulation or maybe it's really not hyperbolic.
- `M.identify()` tells you if the manifold is in the census
- `M.browse()` opens a clickable interface. It'll give you a link diagram if it knows it is a link exterior.
- Cusps
  - `M.cusp_info()` tells you about the cusps and their shape. Numbering starts at 0.
  - Unless $M$ was given as a link exterior from Plink, it will use the two shortest slopes for a basis. Roughly speaking, any non-hyperbolic filling is short.
  - If $M$ is a link exterior from Plink, then its cusps have the standard meridian-longitude basis. So $(1,0)$ is the meridian and $(0,1)$ is the longitude.
- Dehn filling
  - `M.dehn_fill([(a,b), (c,d), (e,f)])` fills the cusps of `M` with the specified slopes. Numbering starts at 0, so the 1st cusp is filled with slope $(c,d)$.
  - Filling with slope $(p,q)$ when $\gcd(p,q)>1$ gives an orbifold. So $(2,0)$ filling a knot complement
  - Use `(0,0)` to keep the cusp unfilled.
- `M.drill(n)` gives the complement of the $n$th shortest geodesic. Numbering starts at 0.
- `M.fundamental_group()` and `M.homology()` give the fundamental group and 1st homology of $M$. These work even if $M$ is not hyperbolic.
- `M.volume()` gives the volume of the hyperbolic manifold $M$.
- `M.symmetry_group()` returns the symmetry group of the manifold $M$.
- `M.is_isometric_to(N)` tells you if the hyperbolic manifolds $M$ and $N$ are isometric.
- `M.is_isometric_to(N, return_isometries=True)` gives information on the action of the symmetries on the cusps

# Basic Notes on KLO

- When drawing, hold Shift to crouch.
- When Processing a diagram, click crossings to correct.
- Focus on component with Ctrl -click, Cmd -click, or right-click. Also hold Shift to add more components to focus.
- Focus on a single component and then click the little colored box to change the color of the component.
- Click on region for Reidemeister I, II, III simplification.
- Drag over/under strand to adjacent crossing.
- Rolfsen Twists (twisting about a circle)
  - Remove self-crossings from unknot component
  - Click disk to eliminate interior crossings
  - P Parallelize strands through circle
  - T Twist about circle, enter number
  - It can be useful to simplifty the diagram before twisting.
- Click $\infty$–framed component to remove it.
- Click $\Delta$ to draw in $\infty$–framed unknots for twisting.
- From the menu, “Convert to Knot” to calculate invariants of the underlying link.