Heegaard Floer homology and Dehn surgery

Problem Set 3

Problem 1. Suppose that $K \subset S^3$, n is a positive integer. Prove that the set

$$\{\mathfrak{s} \in \mathrm{Spin}^c(S^3_K(n)) |\, HF_{\mathrm{red}}(S^3_K(n), \mathfrak{s}) \neq 0\}$$

has at most $2g(K)-1$ elements. In particular, if Y is a rational homology sphere, and there are exactly N Spin^c structures $\mathfrak{s} \in \mathrm{Spin}^c(Y)$ satisfying $HF_{\mathrm{red}}(Y, \mathfrak{s}) \neq 0$, then Y cannot be obtained by integer surgery on any knot in S^3 with genus $\leq \frac{N}{2}$.

Problem 2. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{i < 0, j \geq k\}) \cong \mathbb{Z}\langle 1, U^{-1}, \dots, U^{1-t}\rangle$ for some integer $t \geq 0$.
(2) Prove

$$\chi(C\{i < 0, j \geq k\}) = t_k = \sum_{n=1}^{\infty} n a_{n+k},$$

where a_i's are the coefficients of the normalized Alexander polynomial.
(3) Prove $t = t_k$.

Problem 3. Let $K \subset S^3$ be an L-space knot, $C = CFK^\infty(S^3, K)$, $k \in \mathbb{Z}$.
(1) Prove that $H_*(C\{\max(i, j-k) = 0\}) \cong \mathbb{Z}$.
(2) Prove that $H_*(C\{i < 0, j = k\})$ is either 0 or $\mathbb{Z}$, the same is true for $H_*(C\{i = 0, j \leq k\})$.
(3) Prove that exactly one of the two groups $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i = 0, j \leq k\})$ is $\mathbb{Z}$.
(4) Prove that if $H_*(C\{i = 0, j = k\}) \cong \mathbb{Z}^2$, then both $H_*(C\{i < 0, j = k\})$ and $H_*(C\{i \leq 0, j = k\})$ are $\mathbb{Z}$.
(5) Prove that $H_*(C\{i = 0, j = k\})$ is either 0 or $\mathbb{Z}$. As a consequence, the coefficients of the Alexander polynomial of an L-space knot are 0 or ± 1.

Character Varieties, Surfaces and Applications to Surgery Theory
Problems for Lecture 3

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Exercises for Lecture 3

1. Let $V \cong \mathbb{R}^2$ and $\phi_j : V \to \mathbb{R}$ a non-zero homomorphism for $1 \le j \le n$. Define

$$\| \cdot \| : V \to [0, \infty)$$
$$v \mapsto \sum_j |\phi_j(v)|$$

   (a) Show that $\| \cdot \|$ is a (non-zero) seminorm.

   (b) If $\| \cdot \|$ is a norm show that its ball of radius 1 is a finite-sided polygon each of whose vertices lies on one of the lines $\ker(\phi_j)$.

   (c) Determine the ball of radius 1 of $\| \cdot \|$ when it is not a norm.

2. If M is the trefoil exterior then $\pi_1(M) = \langle \gamma_1, \gamma_2 : \gamma_1^2 = \gamma_2^3 \rangle$. The fundamental group of ∂M is generated by a meridian

$$\mu = \gamma_1 \gamma_2^{-1}$$

and the Seifert fibre class

$$h = \gamma_1^2,$$

which is central in $\pi_1(M)$. We saw in Lecture 2 that $X^{irr}(M)$ is a curve X_0 which has image $\{(0, 1, w) : w \in \mathbb{C}\} \cong \mathbb{C}$ under the map $X_0 \to \mathbb{C}^3, \chi \mapsto (\chi(\gamma_1), \chi(\gamma_2), \chi(\gamma_1\gamma_2))$.

   (a) Show that for $z \in \mathbb{C}$ there is a homomorphism $\rho_z : \pi_1(M) \to SL(2, \mathbb{C})$ given by

$$\rho_z(\gamma_1) = \begin{pmatrix} i & 0 \\ 0 & -i \end{pmatrix} \quad \rho_z(\gamma_2) = \begin{pmatrix} z & -(z^2 - z + 1) \\ 1 & 1 - z \end{pmatrix}$$

   and that the map $z \mapsto \chi_{\rho_z}$ parametrises X_0.

   (b) Show that $\|h\|_{X_0} = 0$ and $\|\mu\|_{X_0} = 2$[1] and deduce that if $\alpha \in H_1(\partial M)$ then

$$\|\alpha\|_{X_0} = 2|\alpha \cdot h|$$

   where $\alpha \cdot h$ is the algebraic intersection of α and h on ∂M.

[1] **Hint**: Since $X_0 \cong \mathbb{C}$ it has a unique ideal point. Recall that $\mu = \gamma_1\gamma_2^{-1}$. Calculate the multiplicity of the ideal point as a pole of $f_\mu : X_0 \to \mathbb{C}$ using the parametrisation of X_0 given above.

0.1 References for Lecture 3

1. M. Culler and P. B. Shalen *Varieties of group representations and splittings of* 3*-manifolds*, Ann. Math. **117** (1983), 109–146.

2. P. B. Shalen, *Representations of 3-manifold groups*, in **Handbook of Geometric Topology**, R. Daverman and R. Sher, eds. North-Holland, Amsterdam, 2002, pp. 955–1044.