Heegaard Floer homology and Dehn surgery

Problem Set 1

Problem 1. Let $\alpha_1, \alpha_2, \dots, \alpha_n$ be n mutually disjoint, simple closed curves on a closed oriented surface Σ. Prove that the homology classes $[\alpha_1], \dots, [\alpha_n] \in H_1(\Sigma)$ are linearly independent if and only if the complement $\Sigma \setminus (\alpha_1 \cup \dots \cup \alpha_n)$ is connected.

Problem 2. Find a genus 1 Heegaard diagram of S^3, and use it to compute $HF^\infty(S^3), HF^-(S^3), HF^+(S^3)$.

Problem 3. Let

$$\big(\Sigma, \{\alpha_1, \dots, \alpha_g\}, \{\beta_1, \dots, \beta_g\}\big)$$

be a Heegaard diagram of Y. Prove

$$H_1(Y) \cong H_1(\Sigma)/\langle [\alpha_1], \dots, [\alpha_g], [\beta_1], \dots, [\beta_g] \rangle.$$

Problem 4. Prove the map $\delta\colon\ \mathrm{Spin}^c(Y) \to H^2(Y)$ is a one-to-one correspondence.

Problem 5. Suppose $\mathfrak{s}_1, \mathfrak{s}_2 \in\ \mathrm{Spin}^c(Y)$, prove

$$\delta(\mathfrak{s}_1, \mathfrak{s}_2) = -\delta(\overline{\mathfrak{s}_2}, \overline{\mathfrak{s}_1}), \quad c_1(\mathfrak{s}_1) - c_1(\mathfrak{s}_2) = 2\delta(\mathfrak{s}_1, \mathfrak{s}_2).$$

As a consequence, show that the map $c_1\colon\ \mathrm{Spin}^c(Y) \to H^2(Y)$ is injective if $H_1(Y)$ has no 2–torsion.

CODIMENSION ONE FOLIATIONS - PROBLEM SET 1
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(1) Let $\mathcal{F}_m$ be the foliation of T^2 described in Lecture 1.

(a) Prove that if $m \in \mathbb{Q} \cup \{\frac{1}{0}\}$, then the leaves of $\mathcal{F}_m$ are simple closed curves.

(b) Prove that if $m \notin \mathbb{Q} \cup \{\frac{1}{0}\}$, then the leaves of $\mathcal{F}_m$ are injectively immersed copies of $\mathbb{R}$, and any leaf of $\mathcal{F}_m$ is dense in T^2.

(c) What can you say about the leaf space of $\mathcal{F}_m$?

(2) An alternate way of viewing $\mathcal{F}_m$: as a *suspension of a homeomorphism of* S^1:
Form a foliation $\mathcal{G}_\phi$ of T^2 as follows. First foliate $[0,1]\times S^1$ by straight line segments $[0,1] \times \{t\}$. Then glue $\{1\} \times S^1$ to $\{0\} \times S^1$ by the homeomorphism

$$\phi : \{1\} \times S^1 \to \{0\} \times S^1 : (1, e^{2\pi i t}) \mapsto (0, e^{2\pi i m t}).$$

Let γ denote the simple closed curve $\{1\} \times S^1$. Revisit the questions of (1a) and (1b) with this model of $(T^2, \mathcal{F}_m)$ in mind.

(3) *Denjoy blow-up* and *Denjoy splitting*:

If necessary, consult Wikipedia on the Cantor function $c : [0,1] \to [0,1]$ before proceeding further. Recall that the Cantor function is continuous. Note that under the identification $S^1 = [0,1]/\sim$, where $0 \sim 1$, there is a "Cantor function" $c : S^1 \to S^1$.

Suppose m is irrational, and view $\mathcal{F}_m$ as given by a suspension of ϕ as given in (2). The *Denjoy blowup* and *Denjoy splitting* of $\mathcal{F}_m$ along a leaf L is described as follows.

Label the leaves of $\mathcal{F}$ by L_x, where $x \in L \cap \gamma$. Pick any $x \in S^1$, and let $x_n = \phi^n(x)$ for all $n \in \mathbb{Z}$. (Equivalently, pick any leaf L of $\mathcal{F}_m$ and enumerate the countable many points $L \cap \gamma$ as $x_n, n \in \mathbb{Z}$.)

Make the following precise:

(a) "Blow up" S^1 to a longer circle C by replacing each x_n by a compact interval J_n, where the sum of the lengths of the intervals J_n is finite.

(b) Define $\psi : C \to C$ to agree with ϕ on the complement of the orbit x_n, and extend linearly over the interior of the intervals J_n.

The foliation $\mathcal{G}_\psi$ is called the *Denjoy blowup* of $\mathcal{F}_m = \mathcal{G}_\phi$ along L_x. Removing the leaves of $\mathcal{G}_\psi$ passing through the interiors of the the J_n results in a foliation $\mathcal{F}'_m$ obtained by *Denjoy splitting* $\mathcal{F}_m$ open along the leaf L_x. Note that the transverse cross-section $\mathcal{F}'_m \cap \gamma$ is a Cantor set.

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(4) Let τ be a train track, and $N(\tau)$ an I-fibered regular neighbourhood of τ. Recall that a curve is *carried by* τ if it can be isotoped to lie in $N(\tau)$ everywhere transverse to the I-fibering. It is *fully carried by* τ if it is carried, with nonempty intersection with each I-fiber. A foliation is carried (respectively, fully carried) by τ if after Denjoy splitting along finitely many leaves, the resulting lamination can be isotoped so that every leaf is carried (fully carried) by τ.

Identify which foliations $\mathcal{F}_m$ are carried (respectively, fully carried) by the train tracks τ shown below. Do these train tracks carry (respectively, fully carry) any other foliations of T^2?