Lines in the tropics

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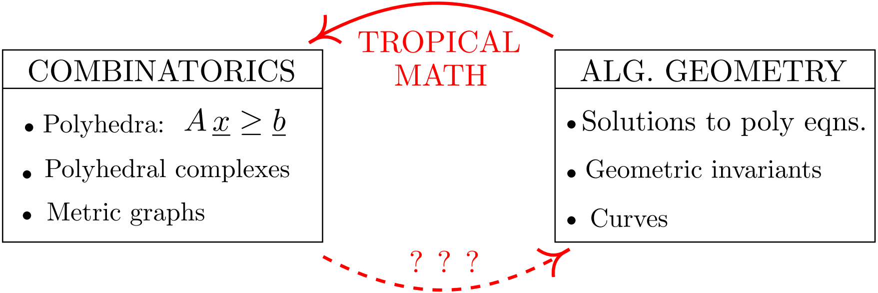
Based on joint works in preparation with Anand Deopurkar (Australia) and Hannah Markwig (Germany)

Tropical Mathematics



SLOGAN 1: Tropical Geometry is **Algebraic Geometry over the tropical semifield** $(\overline{\mathbb{R}}, \oplus, \odot)$.

SLOGAN 2: Tropical varieties are **combinatorial shadows** of algebraic varieties (over valued fields.)

SLOGAN 1: Trop. Geometry is Alg. Geometry over $\overline{\mathbb{R}_{tr}} := (\overline{\mathbb{R}}, \oplus, \odot)$.

- $\overline{\mathbb{R}} = \mathbb{R} \cup \{-\infty\}$, $a \oplus b = \max\{a, b\}$, $a \odot b = a + b$

(E.g.: $3 \oplus 5 = 5$, $3 \odot 5 = 8$, $-\infty \oplus 3 = 3$, $0 \odot 3 = 3$.)

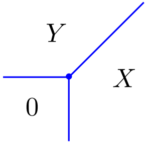
Polys in $\overline{\mathbb{R}_{tr}}[X_1, \ldots, X_n] \equiv \overline{\mathbb{R}}^n \to \mathbb{R}$ cont., convex, affine PL with $\mathbb{Z}$-slopes

$$F(\underline{X}) = \bigoplus_{\substack{\alpha \in \mathbb{N}_0^n \\ \text{(finite)}}} a_\alpha \odot X_1^{\odot \alpha_1} \odot \cdots \odot X_n^{\odot \alpha_n}$$

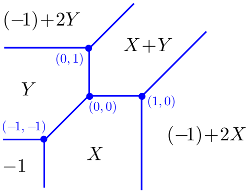
$$= \max_\alpha \{a_\alpha + \alpha_1 X_1 + \ldots + \alpha_n X_n\}$$

- Tropical hypersurface: $V_{\mathrm{tr}}(F)$ = corner locus of F.

- **Examples:**



$F(X,Y) = \max\{0, X, Y\}$



$F(X,Y) := \max\{-1, X, Y, X+Y, -1+2X, -1+2Y\}$

SLOGAN 2: Trop. vars are *comb. shadows* of alg. vars via *valuations*.

- **Def.:** A valuation on a field K is a map $\mathrm{val}\colon K \smallsetminus \{0\} \to \mathbb{R}$ satisfying:
  (1) $\mathrm{val}(xy) = \mathrm{val}(x) + \mathrm{val}(y)$,
  (2) $\mathrm{val}(x+y) \geqslant \min\{\mathrm{val}(x), \mathrm{val}(y)\}$ (and $=$ if $\mathrm{val}(x) \neq \mathrm{val}(y)$)

Extend val to K via $\mathrm{val}(0) = +\infty$.

Examples:
- Trivial valuations $\mathrm{val}(x) = 0$ for all $x \neq 0$.
- $K = \overline{\mathbb{C}((t))}$ with t-valuation ($\mathrm{val}(2t^{-5}+3t^{-1/2}+\ldots)=-5$).
- $K = \overline{\mathbb{Q}_p}$ with p-adic valuation.

- We tropicalize polynomials in $K[x_1, \ldots, x_n]$ using ($-val$ on K, $\oplus$ and $\odot$):

$$f(\underline{x})=\sum_{\alpha} a_\alpha \, \underline{x}^\alpha \rightsquigarrow trop(f)(\underline{X})=\max_{\alpha\in \mathrm{Supp}(f)}\{-\mathrm{val}(a_\alpha)+\alpha_1X_1+\ldots+\alpha_nX_n\}$$

- **Def. 1:** $\mathrm{Trop}(V(f))$ = Corner locus of $\mathrm{trop}(f)$ in $\overline{\mathbb{R}}^n$ (max is at two α's)

In general: I defining ideal $\rightsquigarrow$ $\boxed{\mathrm{Trop}(I) = \bigcap_{f\in I} \mathrm{Trop}(V(f)).}$

SLOGAN 2(cont.): Trop. vars are *comb. shadows* of alg. vars via *valns.*

Fix $K = \overline{K}$ with **non-trivial valn.** (e.g. $K = \overline{\mathbb{C}((t))}$).

Fix a closed embedding $\iota\colon X \hookrightarrow Y_\Sigma =$ toric variety with dense torus $(K^*)^n$.

Examples: $Y_\Sigma = (K^*)^n$, K^n or $\mathbb{P}^n$.

Trop $Y_\Sigma = \mathbb{R}^n$, $\overline{\mathbb{R}}^n$ or $\mathbb{TP}^n := \frac{\overline{\mathbb{R}}^{n+1} \smallsetminus \{(-\infty,\ldots,\infty)\}}{\mathbb{R}\cdot\mathbf{1}} \simeq \Delta_n$ (n-simplex).

Def. 2: Trop $X = \text{cl.}\{(-\text{val}(p_1), \ldots, -\text{val}(p_n)) : (p_1, \ldots, p_n) \in X\} \subset$ Trop Y_Σ

Fundamental Thm. of Trop. Geom.: Both definitions agree.

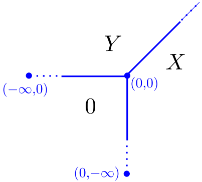
Structure Thm.: Trop(X) is a polyhedral complex of dimension $\dim(X)$ (pure if X is irreducible, balanced if multiplicities on top-dim. cells.)

ISSUE: Definition of Trop(X) is coordinate dependent! (Q: Best choices?)

Examples: Lines in the tropics over $K = \overline{\mathbb{C}((t))}$

- **Example 0:** The line $K \rightsquigarrow \text{Trop}(K) = \overline{\mathbb{R}}$

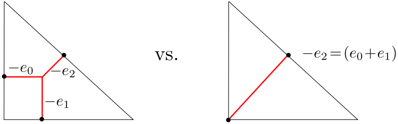
- **Example 1:** The line $1 + x + y = 0$ in the plane K^2.



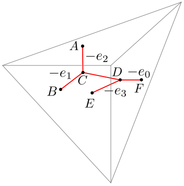
Def. 1: $f = 1 + x + y \rightsquigarrow \text{trop}(f)(X, Y) = \max\{0, X, Y\}$

Def. 2: $\iota\colon K \hookrightarrow K^2 \quad \iota(x) = (x, -1-x) \rightsquigarrow (-\text{val}(x), -\text{val}(1+x))$ in $\overline{\mathbb{R}}^2$

- **Example 2:** Trop. Lines in $\mathbb{TP}^2$



- **Example 3:** Trop. Lines in $\mathbb{TP}^3$



Generic

$A = (0, -1, -\infty, 0)$

$B = (0, -\infty, -2, 0)$

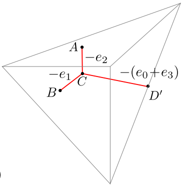
$C = (0, -1, -2, 0)$

$D = (0, 0, -1, 0)$

$E = (0, 0, -1, -\infty)$

$F = (-\infty, 0, -1, 0)$

$D' = (-\infty, -1, -2, -\infty)$

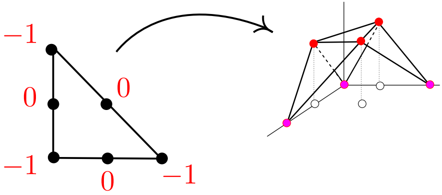


Non-generic

Tropical plane curves = metric graphs in $\mathbb{R}^2$ = dual to Newton subdivisions

Example: $f(x,y) = t + x + y + x\,y + 2t\,x^2 + (3t + t^2)\,y^2$ in $\overline{\mathbb{C}((t))}[x,y]$

$$\text{trop}(f)(X,Y) = \max\{-1, X, Y, X+Y, -1+2X, -1+2Y\}$$



0. Take a polynomial f in $K[x,y]$ with K non-trivially valued field.
1. Build the **Newton Polytope** of f: $NP(f) :=$conv((i,j) in supp(f)).
2. Place each point (i,j) from $NP(f)$ at height $-$val(coeff($x^i y^j$)) in $\mathbb{R}^3$.

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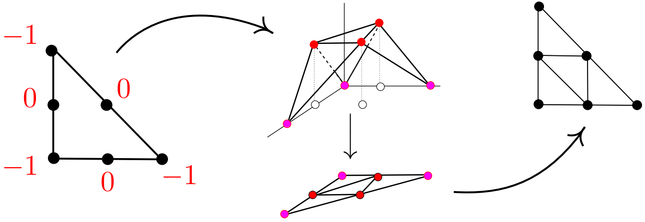

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3. Take upper hull and project to $\mathbb{R}^2$. We get a **subdivision** of $NP(f)$.

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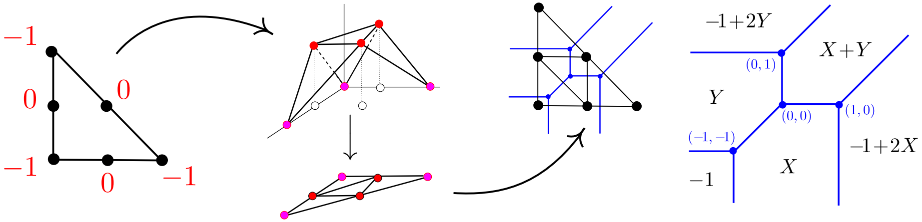

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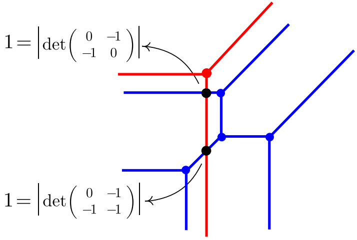
3. Take upper hull and project to $\mathbb{R}^2$. We get a **subdivision** of $NP(f)$.
4. Trop($V(f)$) = dual graph to this subdivision. Comes with a metric.

Basic Facts about general tropical plane curves:

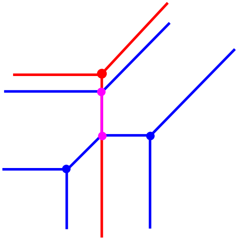
(1) Interpolation for *general* pts in $\mathbb{R}^2$ holds tropically (Mikhalkin's Corresp.)
(unique line through 2 gen. points, unique conic through 5 gen. points,. . .)

(2) *General* trop. curves intersect *properly* and as expected (Trop. Bézout.)



Proper intersection at 2 pts

vs.



Tangent Line

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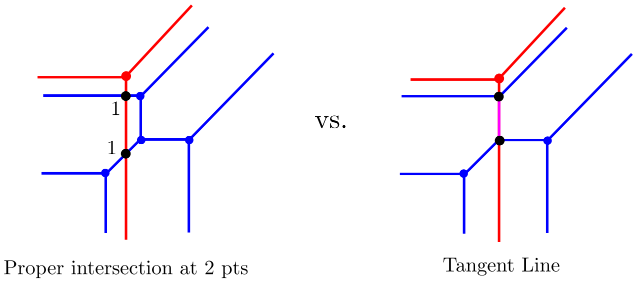
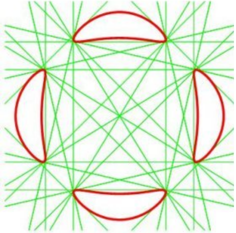


Non-general case: Replace usual intersection with stable intersection.

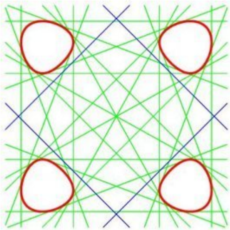
$$C_1 \cap_{st} C_2 := \lim_{\underline{\varepsilon} \to (0,0)} C_1 \cap (C_2 + \underline{\varepsilon}).$$

Today's focus: 2 classical results in Algebraic Geometry

Plücker (1834): A sm. quartic curve in $\mathbb{P}^2_{\mathbb{C}}$ has exactly 28 bitangent lines. (0,4,8,16 or 28 real bitangents, depending on topology of the real curve.)



Trott: 28 totally real bitangents.



Salmon: 28 real, 24 totally real.

Cayley-Salmon (1849): Any smooth algebraic cubic surface in $\mathbb{P}^3_{\mathbb{C}}$ contains exactly 27 distinct lines.

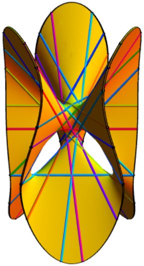


Figure: Clebsch cubic surface

ISSUE: Both results fail tropically! But we can fix it.

28 bitangent lines to sm. plane quartics over $K = \overline{\mathbb{C}((t))}$.

Plücker (1834): A sm. quartic curve in $\mathbb{P}^2_K$ has exactly 28 bitangent lines. (0,4,8,16 or 28 real bitangents, depending on topology of the real curve.)

Question: What happens tropically?

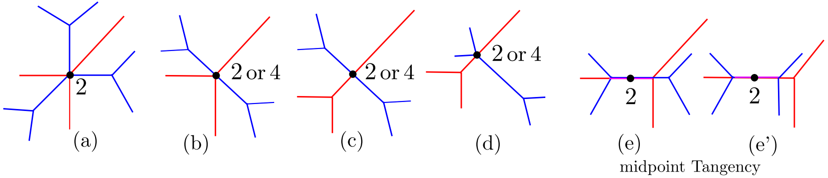
Baker-Len-Morrison-Pflueger-Ren (2015): Every tropical smooth quartic in $\mathbb{R}^2$ has 7 bitangent classes.

Len-Markwig (2017): Generically, each class lifts to 4 classical bitangents.

Len-Jensen (2017): Each class *always* lifts to 4 classical bitangents.

Question: What is a tropical bitangent line? Need tangencies at 2 points.

Len-Markwig: 5 local tangencies (up to $\mathbb{S}_3$-symmetry.)

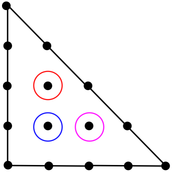


28 bitangent lines to sm. plane quartics over $K = \overline{\mathbb{C}((t))}$.

Theorem: There are 28 classical bitangents to sm. plane quartics over K **but** 7 tropical bitangent classes to their tropicalizations.

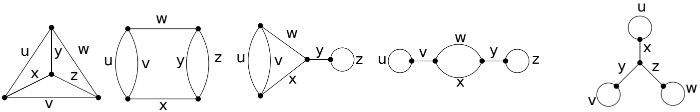
Question: Combinatorial proof?

Trop. sm. quartic = dual to unimodular triangulation of Δ_2 of side length 4.



$\rightsquigarrow$ duality gives a genus 3 planar metric graph.

Possible cases:



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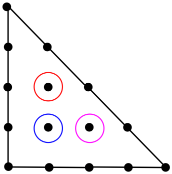

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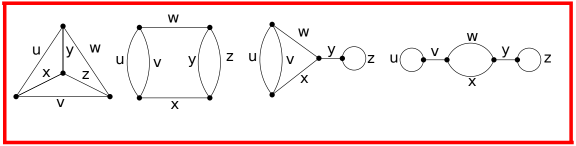
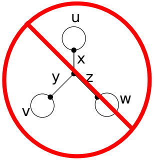
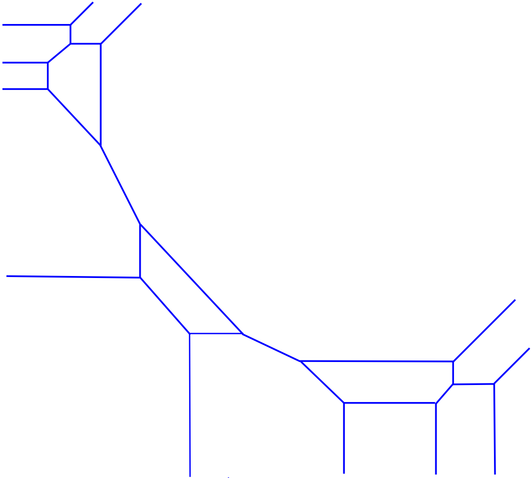
Possible cases: [BLMPR '15]

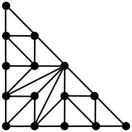




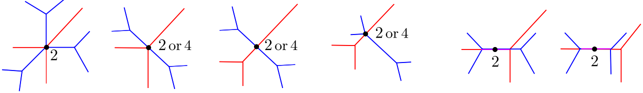
Brodsky-Joswig-Morrison-Sturmfels (2015): Newton subdivisions give linear restrictions on the lengths u, v, w, x, y, z of the edges.

28 classical bitangents vs. 7 tropical bitangent classes.





Local tangencies:
(up to symm.)



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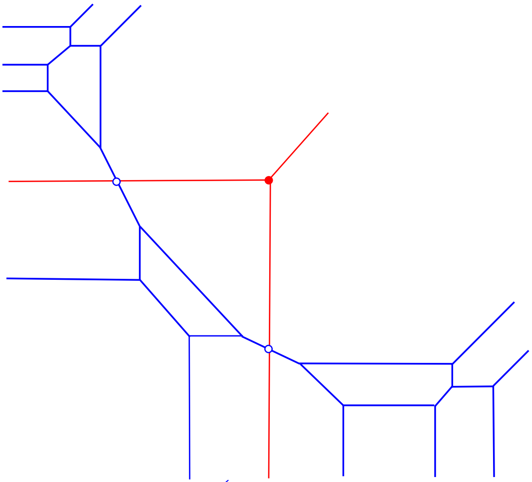

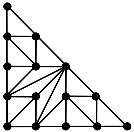

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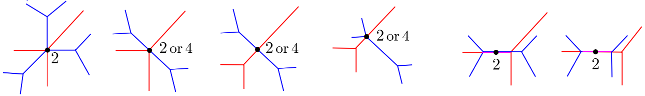

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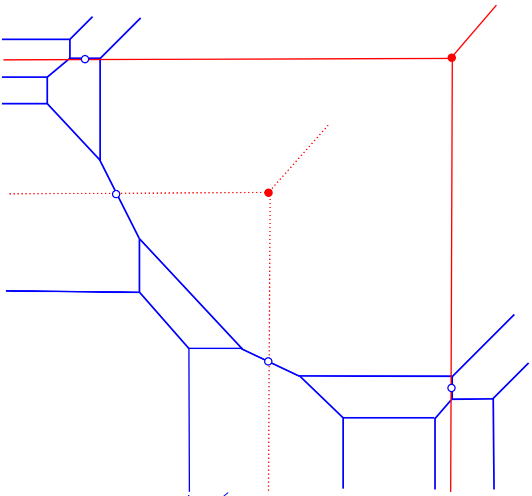

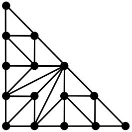

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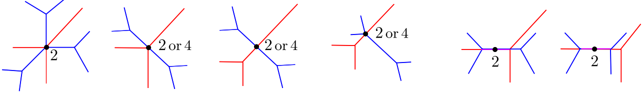

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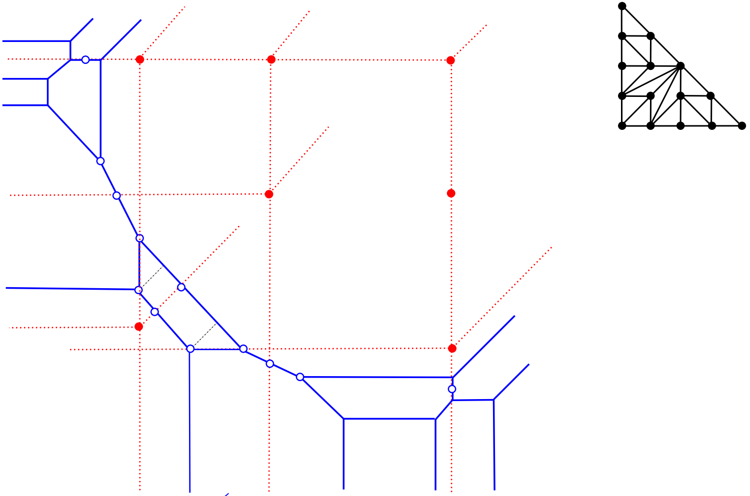

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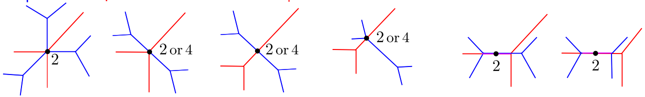

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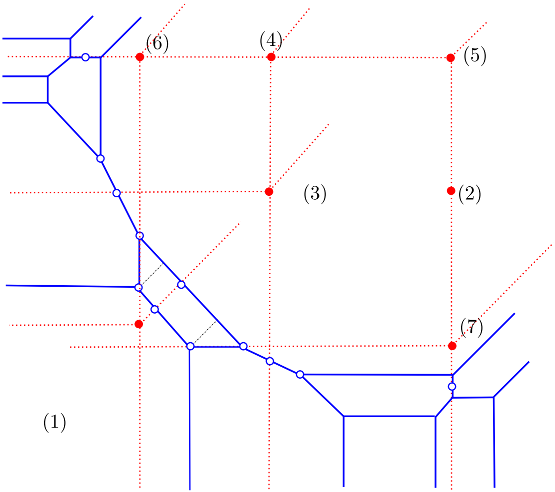

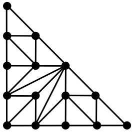

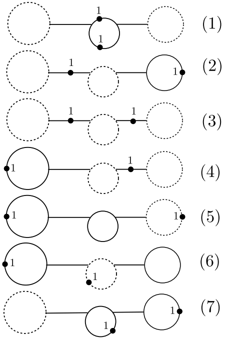

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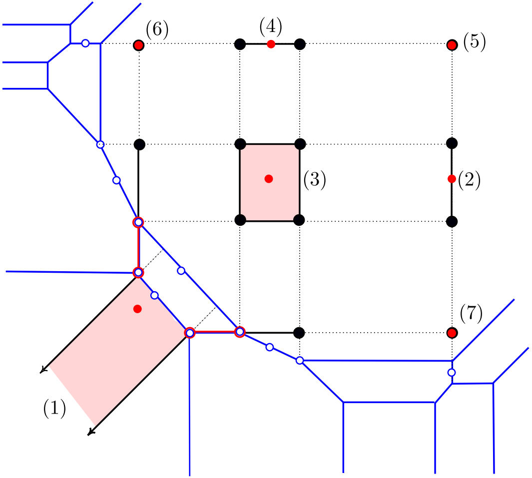

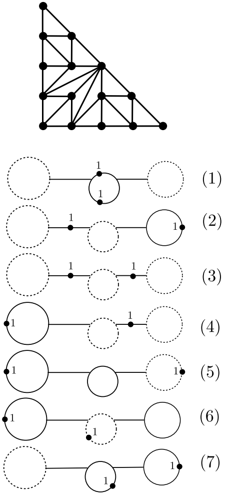

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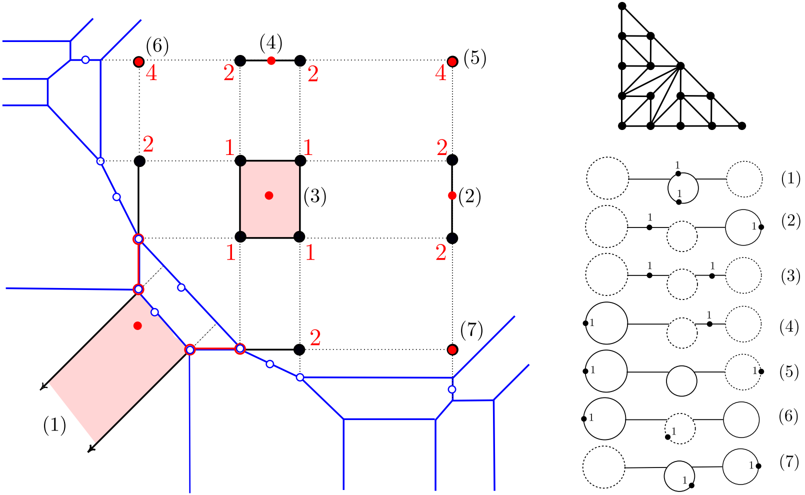


C.-Markwig (2018): There are **36 shapes** of bitangent classes (up to symm.) They are **min-tropical** convex sets. Liftings come from vertices.
Over $\mathbb{R}$: liftings on each shape are either all (totally) real or none is real.

The 27 lines on a sm. cubic surface in $\mathbb{P}^3_K$ for $K = \overline{\mathbb{C}(\!(t)\!)}$

Cayley-Salmon (1849): Any smooth algebraic cubic surface X in $\mathbb{P}^3_K$ contains exactly 27 distinct lines.

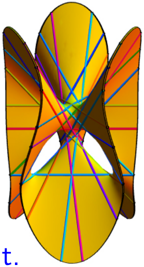


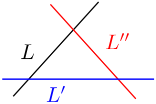
Figure: Clebsch cubic surface

Schläfli, Cayley-Salmon: description of this line arrangement.

- Say L, L' lines of X intersect and let π be the plane in $\mathbb{P}^3_K$ they span.

Then: $X \cap \pi = L \cup L' \cup L''$ and L'' is also a line.

- Generic behavior:



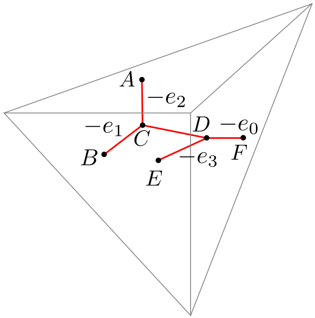
(not concurrent)

- Generically: Every line meets 10 others (which come in 5 pairs).

Gen. dual int. complex: 27 V, 135 E and 45 T = 10-reg. Schläfli graph.

The 27 lines on a sm. cubic surface in $\mathbb{P}^3_K$ for $K = \overline{\mathbb{C}((t))}$

Vigeland (2007): The result fails tropically! He gives examples of Trop X in $\mathbb{TP}^3$ with 1-parameter families of tropical lines (infinitely many lines!)



Generic

$A = (0, -1, -\infty, 0)$

$B = (0, -\infty, -2, 0)$

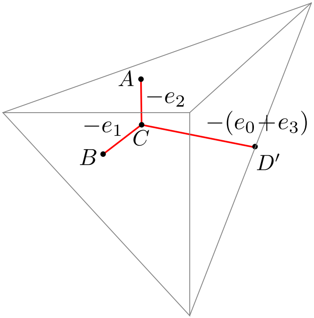
$C = (0, -1, -2, 0)$

$D = (0, 0, -1, 0)$

$E = (0, 0, -1, -\infty)$

$F = (-\infty, 0, -1, 0)$

$D' = (-\infty, -1, -2, -\infty)$



Non-generic

trop. line = in $(\mathbb{TP}^n)^\circ$ balanced metric trees with (r) rays in direction $e_{B_1}, \ldots, e_{B_r}$ with $B_1 \sqcup \ldots \sqcup B_r = \{0, \ldots, n\}$, $e_B := -\sum_{i \in B} e_i$.

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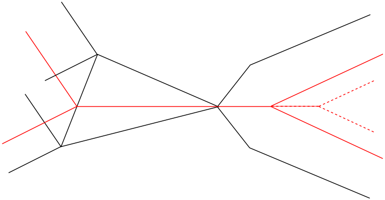


Vigeland (2007), Hampe-Joswig (2016): Combinatorial classification of all tropical cubic surfaces in $\mathbb{TP}^3$ and their lines.

Algebraic approach:

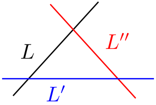
- Cubic surface in $\mathbb{P}^3_K \equiv$ homogeneous degree 3 polynomial in 4 variables.
- 20 coefficients up to global constant, so we get a $\mathbb{P}^{19}_K$ worth of surfaces.
- Smoothness: *open condition* in $\mathbb{P}^{19}_K$.
- Coord. changes give the *same surface*, so we identify points via PGL(4).

Our moduli space of smooth cubic surfaces has dimension $= 20 - 4^2 = \mathbf{4}$.

The 27 lines on a sm. cubic surface in $\mathbb{P}^3_K$ for $K = \overline{\mathbb{C}((t))}$

Running assumption: Our smooth cubics contain **no concurrent lines**.

New approach: Fix the problem by a new embedding (compatible with families), when all the lines are at infinity.

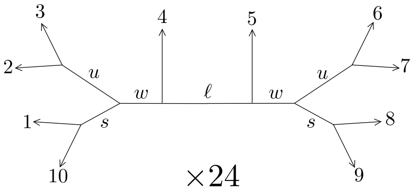


$\rightsquigarrow$ 45 tritangent planes $\rightsquigarrow$ $X \hookrightarrow \mathbb{P}^{44}_K$ "anticanonical embedding"

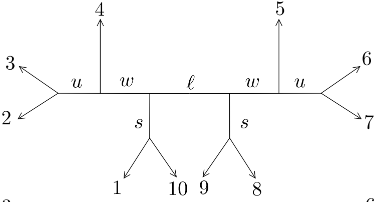
C-Deopurkar (2018): The antican. embedding $X \hookrightarrow \mathbb{P}^{44}_K$ satisfies:

1. Linear span of X in $\mathbb{P}^{44}$ is a $\mathbb{P}^3$ (the original one).
2. All 27 lines of X lie at infinity (on 5 hyperplanes each).
3. Intersections of lines lie in exactly 9 hyperplanes at infinity.
4. Generically, $\operatorname{Trop} X$ in $\mathbb{TP}^{44}$ has exactly 27 lines, all at infinity. Each line is a metric tree with 10 leaves. This arrangement determines $\operatorname{Trop} X$.
5. Otherwise, we have 27 extra lines inside (5 rays each) and $\operatorname{Trop} X$ is a fan. The arrangement at infinity is the 10-reg Schläfli graph (27 V, 135 E.)

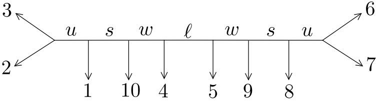
The 27 tropical lines on a gen. trop. cubic surface in $\mathbb{TP}^{44}$



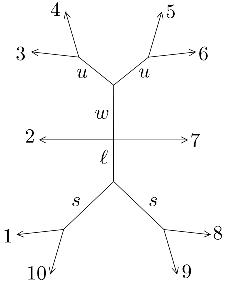
$\times 24$



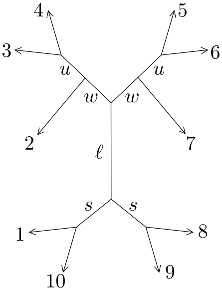
$\times 12$



$\times 12$



$\times 3$



$\times 3$

Type	#cones	Vert.	Edges	Rays	Triangles	Squares	Flaps	Cones
0	1	1	0	27	0	0	0	135
(a)	36	8	13	69	6	0	42	135
(a_2)	270	20	37	108	14	4	81	135
(a_3)	540	37	72	144	24	12	117	135
(a_4)	1620	59	118	177	36	24	150	135
(b)	40	12	21	81	10	0	54	135
(aa_2)	540	23	42	114	13	7	87	135
(aa_3)	1620	43	82	156	22	18	129	135
(aa_4)	540	68	133	195	33	33	168	135
(a_2a_3)	1620	43	82	156	22	18	129	135
(a_2a_4)	810	71	138	201	32	36	174	135
(a_3a_4)	540	68	133	195	33	33	168	135
(ab)	360	26	48	123	16	7	96	135
(a_2b)	1080	45	86	162	24	18	135	135
(a_3b)	1080	69	135	198	34	33	171	135
(aa_2a_3)	3240	46	87	162	21	21	135	135
(aa_2a_4)	1620	74	143	207	31	39	180	135
(aa_3a_4)	1620	74	143	207	31	39	180	135
($a_2a_3a_4$)	1620	74	143	207	31	39	180	135
(aa_2b)	2160	48	91	168	23	21	141	135
(aa_3b)	3240	75	145	210	32	39	183	135
(a_2a_3b)	3240	75	145	210	32	39	183	135
($aa_2a_3a_4$)	3240	77	148	213	30	42	186	135
(aa_2a_3b)	6480	78	150	216	31	42	189	135

Table: recovers Table 1 from [Ren-Shaw-Sturmfels (2016)] for Cox embedding.