A skew product map with a non-contracting iterated monodromy group

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The map $F : F^{-1}(\mathbb{C}^2 \setminus P_F) \longrightarrow \mathbb{C}^2 \setminus P_F$ is a covering map of topological degree 4 of a space by its subset.

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$$\gamma \cdot x_z = x_{\sigma(z)} \cdot \gamma_z$$

for some permutation σ of $f^{-1}(t)$ and a function $(\gamma_z)_{z \in f^{-1}(t)} \in \pi_1(\mathcal{M}, t)^{f^{-1}(t)}$.

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If we take the bundle over the moduli space of the corresponding complex structures on S_2, then the map f induces the associated *skew-product* map.

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$$f_{p_1} : (\hat{\mathbb{C}}, \infty, 1, 0, p_1) \longrightarrow (\hat{\mathbb{C}}, 1, \infty, p_2, 0)$$

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It is obtained starting from a Thurston map with the post-critical portrait

$$* \Longrightarrow x_1 \longrightarrow x_2 \longrightarrow x_3 \Longrightarrow x_4 \longrightarrow x_3$$

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$$\begin{aligned} a_1 &\mapsto \sigma(a_1^{-1}, b_1 a_1), & a_2 &\mapsto \sigma(a_2^{-1}, b_2 a_2), \\ b_1 &\mapsto (1, a_1), & b_2 &\mapsto (1, a_2). \end{aligned}$$

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Since we have to interpret this as a wreath recursion on the fundamental group of the sphere without four points, we have to impose $b_1 a_1 = b_2 a_2$.

We get therefore the recursion ϕ_0:

$$\begin{aligned} a_1 &\mapsto \sigma(a_1^{-1}, b_1 a_1), \\ b_1 &\mapsto (1, a_1), \\ b_2 &\mapsto (1, b_2^{-1} b_1 a_1). \end{aligned}$$

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over the free group.Define also the following recursion ϕ_1:

$$\begin{aligned} a_1 &\mapsto \sigma(1, a_1^{-1} b_1 a_1), \\ b_1 &\mapsto (a_1, 1), \\ b_2 &\mapsto (1, b_2^{-1} b_1 a_1). \end{aligned}$$

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$\phi_1 \circ T(b_2) = (1, b_2^{-1} b_1 a_1)$

We get the recursion

$$
\begin{aligned}
a_1 &\mapsto \sigma(a_1^{-1}b_1^{-1}\ a_1^{-1}\ b_1 a_1, b_1 a_1) = \left((a_1^{-1})^T, (b_1 a_1)^T\right) \\
b_1 &\mapsto (1, a_1^{-1}b_1^{-1}\ a_1\ b_1 a_1) = (1, a_1^T) \\
b_2 &\mapsto (1, b_2^{-1} b_1 a_1) = \left(1, (b_2^{-1} b_1 a_1)^T\right)
\end{aligned}
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showing that $T \cdot \phi_1 = \phi_0 \cdot (T, T)$.

$D \cdot \phi_0$ is

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a_1 &\mapsto \sigma(a_1^{-1} b_1^{-1} b_2, b_1 b_2^{-1} b_1 a_1),\\
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Conjugating the right-hand side by $(1, b_1^{-1}b_2)$, we get

$$a_1 \mapsto \sigma\left((a_1^{-1})^{b_1^{-1}b_2}, b_1a_1^{b_1^{-1}b_2}\right) = \sigma\left((a_1^{-1})^D, (b_1a_1)^D\right)$$
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hence $D \cdot \phi_0 = \phi_0 \cdot (D, Db_2^{-1}b_1)$.

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Similar computations show that $D \cdot \phi_1 = \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1)$. Let's summarize:

$$\begin{aligned} T \cdot \phi_0 &= \phi_1, & T \cdot \phi_1 &= \phi_0 \cdot (T, T), \\ D \cdot \phi_0 &= \phi_0 \cdot (D, D b_2^{-1} b_1), & D \cdot \phi_1 &= \phi_1 \cdot (a_1^{-1}, b_2^{-1} b_1 a_1). \end{aligned}$$

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Taking the "direct sum" $\phi_0 \oplus \phi_1$, we get the following wreath recursion for the iterated monodromy group of F:

$$\begin{aligned} a_1 &= \sigma(a_1^{-1}, b_1 a_1, 1, a_1^{-1} b_1 a_1), \\ b_1 &= (1, a_1, a_1, 1), \\ a_2 &= \sigma(a_2^{-1}, b_2 a_2, a_2^{-1}, b_2 a_2), \\ b_2 &= (1, a_2, 1, a_2), \\ T &= \pi(1, 1, T, T), \\ D &= (D, D b_2^{-1} b_1, a_1^{-1}, a_2), \end{aligned}$$

where $\sigma = (12)(34)$ and $\pi = (13)(24)$.

The limit space?

If a map is locally expanding, then its iterated monodromy group is *contracting* in the sense that the word lengths $\|\cdot\|$ of the coordinates of $\phi^n(g)$ are not more than $\lambda\|g\| + C$ for some constants $\lambda \in (0, 1)$, n, and C.

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In our case we have

$$\left(a_1 a_2^{-1}\right)^N \mapsto \left(\left(b_1 a_1 a_2^{-1} b_2^{-1}\right)^N, \left(a_1^{-1} a_2\right)^N, \left(a_1^{-1} b_1 a_1 a_2^{-1} b_2^{-1}\right)^N, a_2^N\right),$$

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which shows that the wreath recursion is not contracting. Moreover, the iterated monodromy group (i.e., the smallest quotient compatible with the recursion) is not contracting, since a_2 is of infinite order. This means that F is not *sub-hyperbolic*: we can not define an orbifold containing the complement of the post-critical set on which F is locally expanding.

If a map is locally expanding or sub-hyperbolic, then the support of the measure of maximal entropy of the map is uniquely determined by the wreath recursion via the construction of the *limit space*.

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The frames of the movie should correspond to the connected components of the limit space of the subgroup

$$\begin{aligned}
a_1 &= \sigma(a_1^{-1}, b_1 a_1, 1, a_1^{-1} b_1 a_1),\\
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They are well defined in this case, even though the group is not contracting.

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