Deformation spaces of rational functions.

Tanya Firsova

Kansas State University

November 3, 2019

joint work with Jeremy Kahn and Nikita Selinger

Deformation Spaces. Motivation.

- If the parameter space of polynomials or rational functions is higher dimensional, it becomes harder to visualize and study. It is natural to consider curves (varieties) defined by certain postcritical relations. The first such examples were Per_n curves introduced and studied by J.Milnor. He studied the space $\mathcal{M}_2$ of conjugacy classes of quadratic rational maps with marked critical points. He has shown that the space is biholomorphic to $\mathbb{C}^2$. Let c_1, c_2 be critical points of the map f.

$$\mathrm{Per}_n = \{f \in \mathcal{M}_2 : \ f^k(c_1) = c_1 \text{ only for } k = mn, m \in \mathbb{N}\} \tag{1}$$

- Per_n curve is an algebraic curve with punctures.

Period 4 curve.

- Let $f \in \mathcal{M}_2$, assume that the first critical point has orbit of length at least 3.
- By conjugating by a Möbius transformation, we can assume that $c_1 = 0$ is a critical point, $0 \mapsto \infty \mapsto 1$, then f takes the form

$$z \mapsto 1 + \frac{a}{z} + \frac{b}{z^2} =: f_{b,c}(z).$$

- If $f \in \mathrm{Per}_4$, then $0 \to \infty \to 1 \to \rho \to 0$,
- $a = \frac{-\rho^2-\rho+1}{\rho-1}$ and $b = \frac{2\rho^2-\rho}{\rho-1}$ are rational functions in ρ, Per_4 is parametrized by ρ.

$$v_\rho = 1 - \frac{(\rho^2+\rho-1)^2}{4(\rho-1)(2\rho^2-\rho)}.$$

- Per_4 is a Riemann sphere with 4 points removed, $\rho \neq 0, 1, \infty, 1/2$.

Deformation Spaces, An Example.

- A.Epstein suggested deformation space approach to the study of varieties defined by postcritically finite relations. Inspired by the work of W.Thurston, he introduced deformation spaces into holomorphic dynamics.
- $$\mathrm{Per}'_n = \{f \in \mathrm{Per}_n : \ f(c_2) \neq c_1 \ \ldots, f^{\circ(n-1)}(c_1)\} \tag{2}$$
- Per'_n is obtained from Per_n by removing finitely many points.
- Let v be the second critical value

$$\mathrm{Per}'_4 := \{f_\rho \in \mathrm{Per}_4 : \ v \neq 0, 1, \infty, \rho\}.$$

Per'_4 is a sphere with 10 points removed, 2 points each for the solutions of the equation $v = 0$, $v = 1$, $v = \rho$.

Deformation Spaces, An Example

- Let $f \in \mathrm{Per}'_n$, $A = \{c_1, \ldots, f^{\circ(n-1)}(c_1)\}$, $B = A \cup \{f(c_2)\}$. Since $f(A) \subset B$,

$$f : (\hat{\mathbb{C}}, A) \to (\hat{\mathbb{C}}, B).$$

- Since critical values of f are in B, one can lift any complex structure from $(\mathbb{S}^2, B)$ to $(\mathbb{S}^2, A)$. This operation induces a map

$$f^* : \mathrm{Teich}(B) \to \mathrm{Teich}(A)$$

- Since $A \subset B$, the forgetful map

$$i_* : \mathrm{Teich}(B) \to \mathrm{Teich}(A)$$

  is well-defined.
- The set of structures

$$\mathrm{Def}(f, A, B) = \{\tau \in \mathrm{Teich}(B) : \quad f^*(\tau) = i_*(\tau)\}$$

  is an example of A.Epstein's deformation space.

Deformation Spaces, An Example

- Each structure in the deformation space corresponds to a unique conjugacy class of rational maps.
  Let $\tau \in \mathrm{Def}(f, A, B)$. If we choose a representative $\phi : (\mathbb{S}^2, B) \to (\hat{\mathbb{C}}, \phi(B))$ of τ, then there is a map ψ and a rational map g such that the following diagram is commutative:

$$\begin{array}{ccc} (\mathbb{S}^2, A) & \xrightarrow{\psi} & (\hat{\mathbb{C}}, \psi(A)) \\ \downarrow f & & \downarrow f_\tau \\ (\mathbb{S}^2, B) & \xrightarrow{\phi} & (\hat{\mathbb{C}}, \phi(B)) \end{array}$$

  Because $f^*\tau = i_*\tau$, by applying a Möbius transformation to ψ, we can assume $\psi \sim \phi$ rel A. If $\#A \geq 3$, then Möbius transformation is unique, and the class of $[f_\tau] \in \mathrm{Rat}_d$ is well-defined in the sense that it is independent of the choice of ϕ and ψ.

Deformation Spaces, An Example

- The projection of this deformation space to the modular space is a union of some components of Per_n', $n \geq 4$. (It is not known if Per_n curve is irreducible, so there might be several components.)
- In the modular space a rational map $f \in \mathrm{Per}_n'$ is represented by $B = (c_1, f(c_1), ..., f^{\circ(n-1)}(c_1), v)$.
- A map $f \in \mathrm{Per}_4'$ is represented by $(0, 1, \infty, \rho, v)$. $\mathrm{Per}_4' \subset \mathrm{Mod}(4)$.

Deformation Space, Definition

- Let $\mathbb{S}^2$ be a sphere, let $f : \mathbb{S}^2 \to \mathbb{S}^2$ be a branched cover. Let $V(f)$ be the set of critical values. Let A, B be finite sets such that

$$A \subset B \subset \mathbb{S}^2, \quad f(A) \cup V(f) \subset B.$$

$$\text{Def}(f, A, B) := \{\tau \in \text{Teich}(B) : \ f^*\tau = i_*\tau\},$$

  where $f^* : \text{Teich}(B) \to \text{Teich}(A)$ is defined by pulling back the complex structure by f and then erasing $f^{-1}(B) \setminus A$, $i_* : \text{Teich}(B) \to \text{Teich}(A)$ is defined by erasing $B \setminus A$.
- For a postcritically finite map f, let $A = B$ be the postcritical set. By W.Thurston's theorem, if the orbifold of f is hyperbolic, then $Def(f, A, B)$ consists either of one point or it is empty.

A.Epstein's Work

- Building on the work of W.Thurston, A.Epstein introduced deformation spaces into holomorphic dynamics.
- He showed that if f is not a flexible Lattes map, and $\mathrm{Def}(f, A, B)$ is non-empty, then it is a submanifold of $\mathrm{Teich}(B)$ of dimension $\#(B - A)$.
- For a pcf f with a hyperbolic orbifold, $\mathrm{Def}(f, A, B)$ either empty or zero dimensional.
- For $f \in \mathrm{Per}_n'$, the deformation space $\mathrm{Def}(f, A, B)$ is one-dimensional.
- A.Epstein used deformation spaces to prove transversality results in holomorphic dynamics.

Questions

- Describe the deformation space $\mathrm{Def}(f, A, B)$.
- When is $\mathrm{Def}(f, A, B)$ empty if B contains at least one free critical value?

Group of self-equivalences.

We say that $h \in \mathrm{Map}(B)$ is a self-equivalence ("special liftable") for f if

$$\begin{array}{ccc} (\mathbb{S}^2, A) & \xrightarrow{h'} & (\mathbb{S}^2, A) \\ \downarrow f & & \downarrow f \\ (\mathbb{S}^2, B) & \xrightarrow{h} & (\mathbb{S}^2, B) \end{array}$$

where h' is homotopic to h mod A. Let G_f be a group of self-equivalences.

Lemma

If $\mathrm{Def}(f, A, B)$ *is not empty,* G_f *is the group of deck transformations of* $\mathrm{Def}(f, A, B)$.

Self-equivalences, questions

- Let $\pi : \text{Teich}(B) \to \text{Mod}(B)$.
- $M(f, A, B) = \pi(\text{Def}(f, A, B))$.
- $\text{Per}_4' = M(f, A, B)$ for $f \in \text{Per}_4'$.
- Let $i : M(f, A, B) \to \text{Mod}(B)$, $i_* : \pi_1(M(f, A, B)) \to \text{Map}(B)$. Is $\text{Ker}\, i_* = 0$?
- $i_*(\pi_1(M(f, A, B))) = G_0$, where G_0 is a group of self-equivalence that preserve a component. Describe ${G_f}\big/{G_0}$.

Self-equivalences for Per'_4, examples

- ρ going around ρ^1_ρ or ρ^2_ρ.

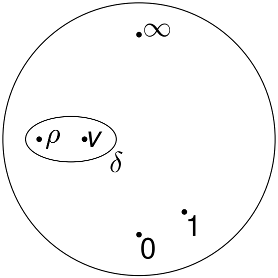


Figure: A curve δ going around ρ and v.

$f^*\delta = i_*(\delta) = 0$. T^2_δ is a self-equivalence.

Equalizing Curves.

- We define analogs of W.Thurston's invariant multicurves.
- Let $\gamma_1, \ldots, \gamma_n$ be simple closed curves. We say that $\Gamma = \{\gamma_i\}$ is a multicurve, if γ_i are pairwise not homotopic and do not intersect.
- Consider free homotopy classes of non-peripheral simple closed curves on $(\mathbb{S}^2, P)$.
- The forgetful map $i : \text{Teich}(B) \to \text{Teich}(A)$ defines

$$i_* : [\gamma]_B \to [\gamma]_A$$

- A formal linear sum

$$m_1\gamma_1 + \cdots + m_n\gamma_n,$$

  where $m_i \in \mathbb{Q} \backslash \{0\}$ is called a weighted multicurve. Let $W(P)$ be a space of weighted multicurves on $(\mathbb{S}^2, P)$.
- The forgetful map extends $i_* : W(B) \to W(A)$.

Thurston's pull-back. Equalizing multicurves.

- Let $\beta_1, \ldots, \beta_k \subset (\mathbb{S}^2, A)$ be non-trivial and non-peripheral preimages of γ under the map f. We first take preimages in $(\mathbb{S}^2, f^{-1}(B))$, then erase the extra punctures and remove trivial and peripheral curves
- Let d_i be the degree of f, restricted to β_i.
- Thurston's pull-back map:

$$f^*(\gamma) = \sum_{i=1}^{k} \frac{1}{d_i} \beta_i.$$

- f^* extends to $f^* : W(B) \to W(A)$.

Equalizing and arithmetically equalizing multicurves.

- We say that a multicurve Γ is *equalizing* if $f^*\Gamma$ and $i_*\Gamma$ are equal as non-weighted multicurves.
- A multicurve Γ is *arithmetically equalizing* if there exists a weighted multicurve $i_*[m_1\gamma_1 \cdots + m_n\gamma_n] = f^*[m_1\gamma_1 + \cdots + m_n\gamma_n]$, $m_1, \ldots, m_n \in \mathbb{Z}\backslash\{0\}$

Dehn Twists around equalizing multicurves.

- Let γ be a simple closed non-peripheral curve. We denote by T_γ a Dehn twist around γ.
- We say that an element $h \in \mathrm{Map}(B)$ is *liftable* under f if there exists $h' \in \mathrm{Map}(A)$ such that

$$f^* h = h' f^*.$$

Lemma

Let $\Gamma = \{\gamma_i\}$ be a multicurve. Assume that a weighed multicurve $m_1\gamma_1 + \cdots + m_n\gamma_n$ (where $m_i \in \mathbb{N} \setminus \{0\}$) is equalizing: $f^*(m_1\gamma_1 + \cdots + m_n\gamma_n) = i_*(m_1\gamma_1 + \ldots m_n\gamma_n)$. Moreover, assume $T_{\gamma_i}^{m_i}$ are liftable. Then $T_{\gamma_1}^{m_1} \circ \cdots \circ T_{\gamma_n}^{m_n}$ is a self-equivalence.

Examples of equalizing multicurve for Per_4' deformation space

Any curve that goes around $(v, 0)$, or $(v, 1)$, or (v, ρ) is an arithmetically equalizing curve.

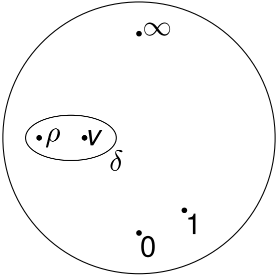
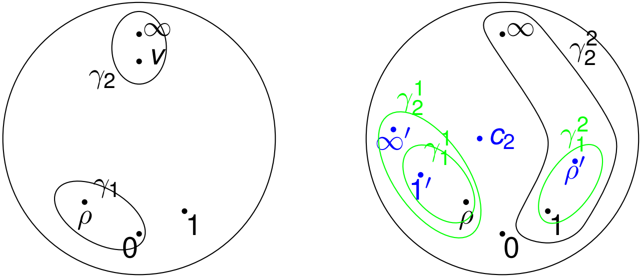


Figure: A curve δ going around ρ and v.

$f^*\delta = i_*(\delta) = 0$. T_δ^2 is a self-equivalence.

Examples of equalizing curves for Per'_4

- Consider a loop ρ around 0. If $\rho \to 0$, then $v \to \infty$.

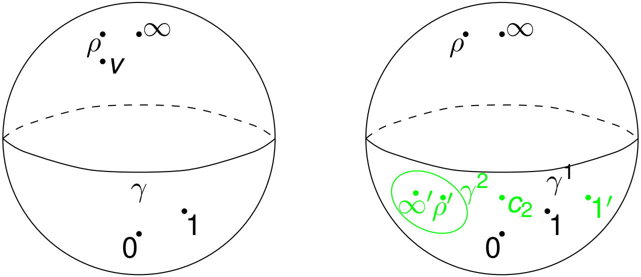


$f^{-1}\gamma_1 = \gamma_1^1 \cup \gamma_1^2$, $f^{-1}\gamma_2 = \gamma_2^1 \cup \gamma_2^2$,
$f^*(\gamma_1) = \gamma_2^2$, $i_*(\gamma_1) = 0$ $f^*(\gamma_2) = 0$, $i_*(\gamma_2) = \gamma_2^2$.

$$f^*(\gamma_1 + \gamma_2) = i_*(\gamma_1 + \gamma_2)$$

$T_{\gamma_1} \circ T_{\gamma_2}$ is a self-equivalence.

Examples of equalizing curves for Per'_4



If $\rho \to \infty$, $v_2 \sim -\frac{\rho}{8}$.
$f^*\gamma = \gamma^1$, $i_*\gamma = \gamma^1$

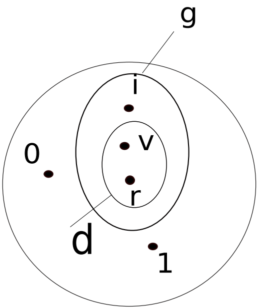
$$f^*\gamma = i_*(\gamma)$$

T_γ is a self-equivalence.

Lemma

If there exists a pair of disjoint arithmetically equalizing curves, then the deformation space is not contractible.

A pair of non-intersecting curves for Per'_4.
Take δ to be a curve going around v and ρ disjoint from γ.



Non-contractibility of the deformation space

- $T_\gamma, T_\delta^2 \in G_f$ and they commute.
- The deformation space of $f \in \mathrm{Per}_4'$ is not contractible.
- E.Hironaka, S.Koch: The deformation space of $f \in \mathrm{Per}_4'$ is not connected.
- Analogous pair of curves γ, δ exist for a deformation space of $f \in \mathrm{Per}_n'$, hence $\mathrm{Def}(f, A, B)$ is not contractible.

Equalizing multicurves and the boundary behavior of the Deformation Space

We denote the stratum in the augmented Teichmuller space obtained by pinching a multicurve Γ by S_Γ.

Theorem

$f^* S_\Gamma = i_* S_\Gamma$ if and only if Γ is an equalizing multicurve. If the deformation space accumulates to S_Γ, then there exists $\tilde{\Gamma} \subset \Gamma$ such that $\tilde{\Gamma}$ is an arithmetically equalizing multicurve.

Vanishing arithmetically equalizing curves

Let n be the dimension of the deformation space $\mathrm{Def}(f, A, B)$. Let $\Gamma = \{\gamma_1, \dots, \gamma_n\}$ be a multicurve on $(\mathbb{S}^2, B)$ consisting of exactly n curves. We denote by f_Γ^* the extension of f^* to the stratum S_Γ. Assume that $f^*\Gamma = i_*\Gamma = \emptyset$.

Lemma

The surface obtained by pinching the multicurve Γ on $(\mathbb{S}^2, B)$ has only one component R_Γ with more than three special points.

Corollary

The map $f : (\mathbb{S}^2, B) \to (\mathbb{S}^2, B)$ induces a postcritically finite map $f_\Gamma : R_\Gamma \to R_\Gamma$.

Vanishing arithmetically equalizing curves

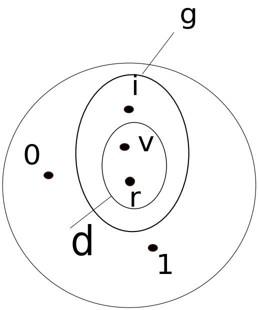
Theorem

Assume that $\Gamma =< \gamma_1, \ldots, \gamma_n >$ *is an equalizing multicurve for an n-dimensional deformation space* $\mathrm{Def}(f, A, B)$ *such that* $f^*\Gamma = i_*\Gamma = \emptyset$*. Assume that the postcritically finite map* f_Γ *has a hyperbolic orbifold. Then the deformation space* $\mathrm{Def}(f, A, B)$ *accumulates to a point on the stratum* S_Γ *if and only if* f_Γ *is Thurston equivalent to a rational map.*

Corollary

Let f *be a branched cover with exactly two simple critical points. Assume that the orbit of the first critical point* c_1 *is periodic and the second critical value* v *does not belong to the forward orbit of* c_1*. Let* A *denote the forward orbit of* c_1 *and* $B = A \cup \{v\}$*. Then* $\mathrm{Def}(f, A, B)$ *is not empty.*

En example of an arithmetically equalizing curve without accumulation to the stratum



- The curve γ is an obstruction for the postcritically finite map $f_\delta : R_\delta \to R_\delta$.
- Deformation space does not accumulate to S_δ.