# Monodromy and Real Wronskians

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# Parametric curves and Wronskians

- Parametric curve $\phi : \mathbb{P}^1 \to \mathbb{P}^k$:

$$t \mapsto \phi(t) = [\ f_0(t) : \cdots : f_k(t)\ ], \text{ where } f_i(t) \in \mathbb{C}[t]_{\leq n}.$$

- The **Wronskian** of $f_0, \ldots, f_k$ is given by

$$\mathrm{Wr}(f_0, \ldots, f_k) = \det \begin{bmatrix} f_0(t) & \cdots & f_k(t) \\ f_0'(t) & \cdots & f_k'(t) \\ \vdots & \ddots & \vdots \\ f_0^{(k)}(t) & \cdots & f_k^{(k)}(t) \end{bmatrix}$$

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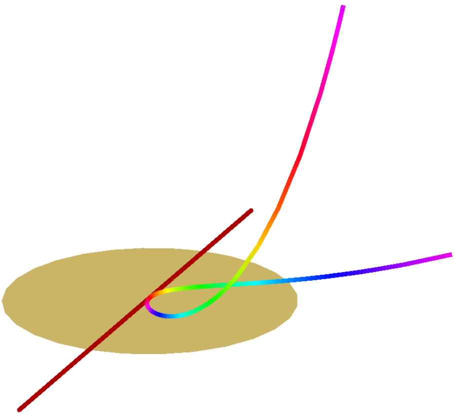
- Detects **flexes**: $t$ such that $\phi, \phi', \phi'', \ldots, \phi^{(k)}$ is linearly dependent (e.g. inflection point, cusp, ...)
  - **Simple flex**: Rank deficiency at $\phi^{(k)}$, fixed at $\phi^{(k+1)}$.

# What is a simple flex in $\mathbb{P}^3$?



- $C$ meets its tangent line to order 2 (not special)
- $C$ meets its tangent plane to order $3 + \mathbf{1} = \mathbf{4}$ (flex!)

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Deep connection to Schubert calculus:

The number of such $\phi$ (counted with multiplicity) is the number of **standard Young tableaux**:

$$\mathrm{SYT}(\begin{array}{|c|c|c|}\hline \ & \ & \ \\ \hline \ & \ & \ \\ \hline\end{array}) = \left\{ \begin{array}{|c|c|c|}\hline 1 & 2 & 4 \\ \hline 3 & 5 & 6 \\ \hline\end{array}\, , \; \begin{array}{|c|c|c|}\hline 1 & 3 & 5 \\ \hline 2 & 4 & 6 \\ \hline\end{array}\, , \dots \right\}$$

# Over $\mathbb{R}$, things are remarkably nice!

Shapiro–Shapiro Conjecture ('95) / M–T–V Theorem:

**Theorem (Mukhin–Tarasov–Varchenko '05, '09)**

*If $\mathrm{Wr}(\phi)$ has all real roots, then $\phi$ itself is defined over $\mathbb{R}$ (up to coordinate change on $\mathbb{P}^k$).*

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**Theorem (L–Purbhoo '19)**

*Let $\mathrm{Wr}(\phi)$ have $n_1$ distinct real roots, $n_2$ complex conjugate pairs.*

*Over $\mathbb{R}$, the number of such $\phi$, counted with signs, is the symmetric group character $\chi^{(3,3)}(2^{n_2}, 1^{n_1})$.*

Recovers M–T–V in the case $n_2 = 0$.

# The Wronski map $\mathrm{Gr} \to \mathbb{P}^N$

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Gives the **Wronski map**:

$$\begin{aligned} \mathrm{Wr} : \mathrm{Gr}(k+1, \mathbb{C}[t]_{\leq n}) &\to \mathbb{P}(\mathbb{C}[t]_{\leq (k+1)(n-k)}), \\ \langle f_0, \ldots, f_k \rangle &\mapsto \langle \mathrm{Wr}(f_0, \ldots, f_k) \rangle. \end{aligned}$$

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Aside #1: Fibers are intersections of Schubert varieties.

# Shapiro–Shapiro / M–T–V, geometric statement

**Theorem (Mukhin–Tarasov–Varchenko '05, '09)**

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- The Wronski map is a **covering map** over the locus of distinct real roots:
  - $\mathrm{UC}_N(\mathbb{RP}^1) :=$ {sets of $n$ distinct points on $\mathbb{RP}^1$} $\subseteq \mathbb{P}(\mathbb{R}[t]_{\leq n})$.

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- Fiber cardinality is *exactly* $\#\mathrm{SYT}(\square\square\square)$
  - Each $\phi$ is **canonically identified** by a tableau [Purbhoo '09].

# Configuration spaces of $\mathbb{RP}^1$ and $\mathbb{CP}^1$

- $\mathrm{UC}_N(\mathbb{RP}^1)$ is much simpler than $\mathrm{UC}_N(\mathbb{CP}^1)$:
  - Fundamental group $\pi_1(\mathrm{UC}_N(\mathbb{RP}^1)) \cong \mathbb{Z}$ by rotation by $\frac{2\pi}{N}$.
  - Open subset $\mathrm{UC}_N(\mathbb{R})$ is a simplex.

# Labeling fibers and monodromy

- Purbhoo '09: Over $\mathrm{UC}_N(\mathbb{R})$, label sheets by tableaux.
  - Label a "limit fiber" near "$\{0, 0, \ldots, 0\}$" $\notin \mathrm{UC}_N(\mathbb{R})$.
  - Orders of vanishing of Plücker coordinates $\rightsquigarrow$ $\begin{array}{|c|c|c|}\hline 1 & 3 & 4 \\ \hline 2 & 5 & 6 \\ \hline \end{array}$

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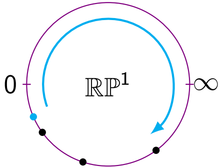


"tableau promotion"

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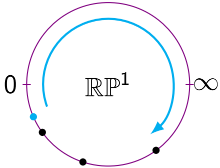


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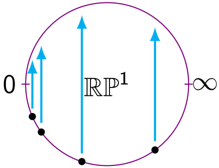


"tableau evacuation"

Aside #2:

- Parallel story over $\overline{M_{0,N}}(\mathbb{R})$ (Kamnitzer, Speyer, Rybnikov)
- Topology, genus of curves in $\mathrm{Gr}(k+1, n+1)$ (L, Gillespie–L)
- Orthogonal Grassmannians (Purbhoo, Gillespie–L–Purbhoo)

# A challenge and a new approach

**Theorem (Mukhin–Tarasov–Varchenko '05, '09)**

*If* $\mathrm{Wr}(\phi)$ *has all real roots, then* $\phi$ *is defined over* $\mathbb{R}$ *(up to change of coordinates).*

**Challenge for geometers**:

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**Now**: conjugate roots in $\mathbb{C}$ and a topological approach.

(−)

| | | |
|---|---|---|
| 1 | 2 | 5 |
| 3 | 4 | 6 |

, (+)

| | | |
|---|---|---|
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, …

Oriented Young tableaux.

# Generalization: complex conjugate roots for $\mathrm{Wr}(\phi)$

**Definition (Cutting up $\mathbb{R}[t]_{\leq N}$)**

For a partition $\mu = (2^{n_2}, 1^{n_1})$, let $P(\mu)$ be

$$P(\mu) = \left\{ \text{polynomials with } \begin{array}{l} n_1 \text{ distinct real roots,} \\ n_2 \text{ complex conjugate pairs} \end{array} \right\}$$
$$\subseteq \mathbb{R}[t]_{\leq N}.$$

Base case: $\mu = (1^N)$, all real roots.

Look at fibers of the restricted Wronski map:

$$\mathrm{Wr}_\mu : \mathrm{Wr}^{-1}(P(\mu)) \to P(\mu).$$

(Note: no roots at $\infty$.)

# Topological and algebraic degrees

**How many real points in the fiber of $\mathrm{Wr}_\mu$?**

- Upper bound from **(algebraic) degree** $= \#\mathrm{SYT}(\begin{smallmatrix}\square&\square&\square\\\square&\square&\square\end{smallmatrix})$.

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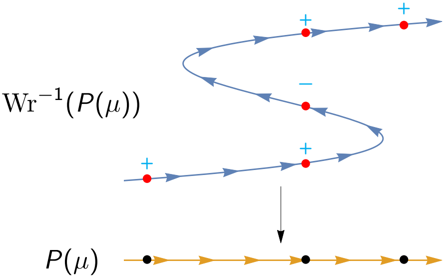


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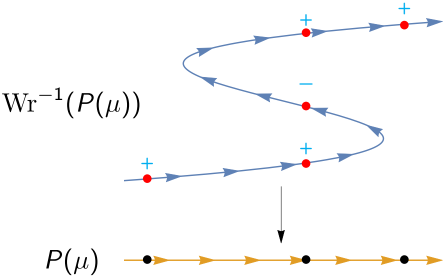


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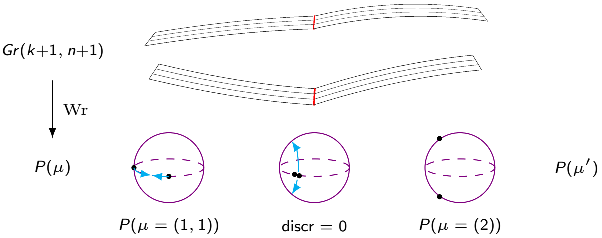
- We use a new **"character" orientation** on the Schubert cell.

# Character orientation of $\mathrm{Gr}(k+1, n+1)$

- Boundary between different $P(\mu)$'s when roots collide:

$$\Delta = \{\mathrm{discr}(\mathrm{Wr}) = 0\} \subset \mathbb{R}[t]_{\leq N}.$$



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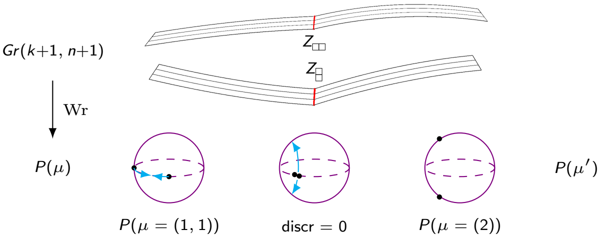


- $\Delta$ lifts to $Z_{\square\square} \cup Z_{\substack{\square\\\square}}$, two kinds of "double flex":
  - Type $\substack{\square\\\square}$: rank deficiency at $\phi^{(k-1)}$ rather than $\phi^{(k)}$.
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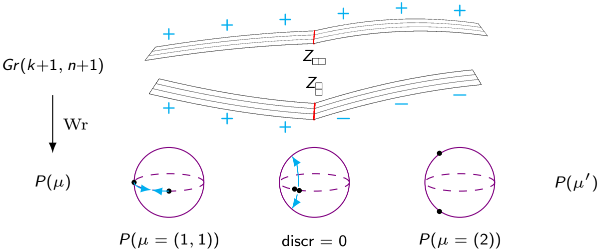


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- **Character orientation**: multiply by the equation of $Z_{\substack{\square\\\square}}$.

# The topological degree of $\mathrm{Wr}_\mu$

**Theorem (L, Purbhoo '19)**

*Under the character orientation, the topological degree of* $\mathrm{Wr}_\mu$ *is the* **symmetric group character** $\chi^{(3,3)}(\mu)$.

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Murnaghan–Nakayama rule: $\mu = (2^{n_2}, 1^{n_1})$, shape $= \lambda$:

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- Special case: $\mu = (1^N)$, no dominos $\rightsquigarrow \chi^{\square\square\square}(1^N) = \#\mathrm{SYT}$.
- **Corollary**: Shapiro–Shapiro Conjecture.

# Labeling fibers by signed Young tableaux

**Proof sketch (geometric Murnaghan–Nakayama rule)**:

- Use the limit fibers labeled by tableaux.
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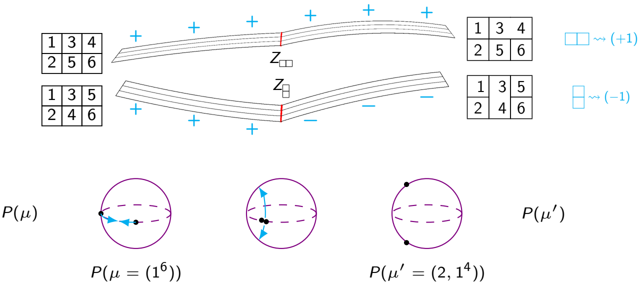
Case 1: $\begin{smallmatrix}3\\4\end{smallmatrix} \longleftrightarrow \begin{smallmatrix}3\\4\end{smallmatrix}$ / $\begin{smallmatrix}3&4\end{smallmatrix} \longleftrightarrow \begin{smallmatrix}3&4\end{smallmatrix}$



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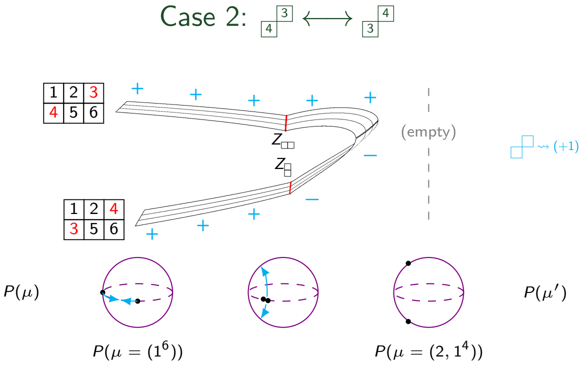

# Open questions

- (Representation theory).
  Do all $S_N$ character values $\chi^\lambda(\mu)$ give topological degrees of real Schubert problems? ($\mu \neq (2^a 1^b)$)

- (Complex geometry).
  Explicit geometry of $\mathrm{Wr}_\mu$ over $P(\mu)$ for $\mu \neq (1^N)$?

- (Stable curves).
  How does the geometry look (for $\mu \neq (1^N)$) over $\overline{\mathcal{M}_{0,N}}$?
  - $\overline{\mathcal{M}_{0,N}}(\mathbb{R})$ is non-orientable!

Many interesting relationships to find.

Thank you!