Lightning Talks
June 2, 2020
Session I

2:15 - 2:20	Caleb Springer, Penn State
2:20 - 2:25	Jacob Mayle, University of Illinois at Chicago
2:25 - 2:30	Pip Goodman, University of Bristol
2:30 - 2:35	Jeroen Hanselman, Universität Ulm
2:35 -2:40	Oana Adascalitei, Boston University
2:40 - 2:45	Vishal Arul, MIT
2:45 - 2:50	Emre Sertöz, Leibniz University Hannover

The Structure of the Group of Rational Points of an Abelian Variety over a Finite Field

Caleb Springer
The Pennsylvania State University

June 2, 2020

Background

The Goal

Given an abelian variety A defined over $\mathbb{F}_q$, recognize the group of rational points $A(\mathbb{F}_q)$ as a **module** over the endomorphism ring $\mathrm{End}_{\mathbb{F}_q}(A)$.

- Lenstra solved this problem completely for **elliptic curves**.

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- Lenstra solved this problem completely for **elliptic curves**.
- In the same paper, Lenstra showed that his result does *not* immediately generalize to all **principally polarized ordinary** abelian varieties.

What we still want:

A generalization of Lenstra's theorem that is true, assuming some conditions that are automatic for elliptic curves.

Main Result

Fix $g \geq 1$. Let $A/\mathbb{F}_q$ be simple of dimension g with Frobenius π. Write $R = \mathrm{End}_{\mathbb{F}_q}(A)$, and let Z be the center of R.

(a) If $[\mathbb{Q}(\pi) : \mathbb{Q}] = 2g$ and R is a **Gorenstein ring**, then

$$A(\mathbb{F}_{q^n}) \cong R/R(\pi^n - 1).$$

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$$A(\mathbb{F}_{q^n}) \cong R/R(\pi^n - 1).$$

(b) If $(\pi^n - 1)Z$ is the product of **invertible prime ideals** in Z, then there is an isomorphism of Z-modules

$$A(\mathbb{F}_{q^n}) \cong (Z/Z(\pi^n - 1))^d.$$

where $d = 2g/[K : \mathbb{Q}]$. The R-module structure comes from an isomorphism of rings

$$R/R(\pi^n - 1) \cong \mathrm{Mat}_d(Z/Z(\pi^n - 1)).$$

Rigidity in Elliptic Curve Local-Global Principles

Jacob Mayle

June 2, 2020

University of Illinois at Chicago

Workshop on Arithmetic Geometry, Number Theory, and Computation

Elliptic curve local-global principles

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Theorem (Katz 1981). If $\delta(\mathcal{T}_\ell) = 1$, then E is K-isogenous to an elliptic curve with nontrivial K-rational ℓ-torsion.

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Question. If $\delta(\mathcal{T}_\ell) \neq 1$, then how large may $\delta(\mathcal{T}_\ell)$ be? Similarly for $\delta(\mathcal{I}_\ell)$.

Rigidity of the locally everywhere conditions

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This rigidity differentiates the local-global principles of Katz & Sutherland with, for instance, the Hasse-Minkowski theorem where failures are quite limited.

References

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- N.M. Katz, *Galois properties of torsion points on abelian varieties*, Invent. Math. **62** (1981), no. 3, 481–502.
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Thank you!

Superelliptic curves with large Galois images

Pip Goodman

University of Bristol

2nd June 2020

Notation

- r a prime
- $f \in \mathbb{Q}[x]$ a polynomial without repeated roots
- C superelliptic curve associated to the smooth affine model $y^r = f(x)$
- J the jacobian of C

Theorem (G.'20)

Suppose $2r|d$ can be written as the sum of two primes $q_1 < q_2$ and there exists a prime $q_2 + 2 < q_3 < d$.
Then we may construct an explicit polynomial $f \in \mathbb{Q}[x]$ of degree d such that for all primes l outside of a finite explicit set the image of the representation

$$\rho_l : G_{\mathbb{Q}} \to \mathrm{Aut}(J[l])$$

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Outline/what's different

- Restrict to
$$\rho_l : G_{\mathbb{Q}(\zeta_{rl})} \to \mathrm{Aut}(J[l]).$$
- The image of ρ_l on this subgroup then lands in the centraliser of $[\zeta_r]$ in $\mathrm{Sp}_{2g}(l)$.
- Let t be the number of primes above l in $\mathbb{Q}(\zeta_r)$. If i the inertia degree of a prime above l in $\mathbb{Q}(\zeta_r)$ is odd, then
$$C_{\mathrm{Sp}_{2g}(l)}(\zeta_r) \cong \mathrm{GL}_{a_1}(l^i) \times \cdots \times \mathrm{GL}_{a_{t/2}}(l^i),$$
otherwise
$$C_{\mathrm{Sp}_{2g}(l)}(\zeta_r) \cong \mathrm{GU}_{a_1}(l^{i/2}) \times \cdots \times \mathrm{GU}_{a_t}(l^{i/2}),$$
where $a_j = a_k$ for any j, k.
- Classify maximal subgroups of $\mathrm{GL}_n(l^i)$ and $\mathrm{GU}_n(l^{i/2})$ containing a "generalised transvection".
- Unlike almost all other Galois image papers, we **do not use/need transvections**.
- Control of inertia groups away from l using T. Dokchitser's "Models of curves over DVRs".
- New method for primitivity which **does not require restrictions** on the ground field.

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The endomorphism character

- $E = \mathrm{End}_K^0(A)$ acts on $V_l(A)$. Thus $V_l(A)$ is an $E \otimes \mathbb{Q}_l$-module.
- The decomposition $E \otimes \mathbb{Q}_l \cong \prod_{\lambda|l} E_\lambda$ induces $V_l \cong \prod_{\lambda|l} V_\lambda$.
- The action of E commutes with of G_K, giving representations

$$\rho_\lambda : G_K \to \mathrm{Aut}_{E_\lambda}(V_\lambda).$$

- The $(\det \circ \rho_\lambda)_\lambda$ form a strictly compatible system of abelian λ-adic representations.
- By work of Ribet they arise from an algebraic Hecke Character Ω, which we call **the endomorphism character** of A with respect to E.

Theorem (G.'20)

Let $\mathfrak{p}$ be a prime of good reduction for $J/\mathbb{Q}(\zeta_r)$ with residual degree one. Suppose $Z\left(\mathrm{End}^0(J_\mathfrak{p})\right)$ is a field and $\mathbb{Q}(\zeta_r) \hookrightarrow Z\left(\mathrm{End}^0(J_\mathfrak{p})\right)$. Then the infinity type of Ω is described by the Newton polygon of C.

- One may often find such a prime by computation.
- Costa, Lombardo and Voight have shown such a primes exists if the Mumford-Tate conjecture holds for $J/\mathbb{Q}(\zeta_r)$.
- If $2r|\deg(f)$, the Mumford-Tate conjecture for $J/\mathbb{Q}(\zeta_r)$ holds by work of Vasiu.

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Gluing curves along their 2-torsion

Let X, Y be (nice) curves of genus 1 and genus 2 over a base field k.

Goal:

Find a curve Z (if it exists) over k such that there is an isogeny:

$$\phi : \mathrm{Jac}(X) \times \mathrm{Jac}(Y) \to \mathrm{Jac}(Z)$$

with $\ker\phi \subset (\mathrm{Jac}(X) \times \mathrm{Jac}(Y))\,[2]$.

Results

- Explicit description of gluing data, which gives: Necessary conditions for when a gluing exists over k.
- Analytic Algorithm $\Rightarrow$ Construct period matrix of $\mathrm{Jac}(Z)$ and reconstruct curve using algorithm by Lercier, Ritzenthaler,Sijsling.
- Formula using interpolation that uses X, Y and gluing data as input.

Results

- Geometric Algorithm

$$\begin{array}{ccc} Z = X \times_{\mathrm{Kum}(Y)} \mathrm{Jac}(Y) & \dashrightarrow & \mathrm{Jac}(Y) \\ \downarrow \overline{\pi} & & \downarrow \pi \\ X & \dashrightarrow & \mathrm{Kum}(Y) \end{array} .$$

- Reverses construction by Ritzenthaler, Romagny.

Above methods described in (H., Schiavone, Sijsling)

Rational Points of Fermat Quartics

Oana Adascalitei

Boston University

June 2, 2020

Introduction

In the 1990s J.-P. Serre challenged the mathematical community to find all the rational points of the Fermat quartic which has the affine model

$$\mathcal{F} : x^4 + y^4 = 17.$$

Are there any other solutions beyond

$$\{(\pm 2, \pm 1), (\pm 1, \pm 2)\}?$$

Would any classical method work?

- Chabauty-Coleman: $\mathrm{genus}(\mathcal{F}) = 3$, $\mathrm{rank}(\mathrm{Jac}(\mathcal{F})) = 6$.
- Manin-Demjanenko:

$$\mathrm{Jac}(\mathcal{F}) \sim E_1 \times E_1 \times E_2,$$

where $\mathrm{rank}(E_1) = \mathrm{rank}(E_2) = 2$.

Flynn-Wetherell

In early 2000s, V. Flynn and J. Wetherell developed techniques to address this challenging problem. Their strategy employed a cover for their original curve, which they combined with the elliptic Chabauty method.

Theorem (Flynn, Wetherell)

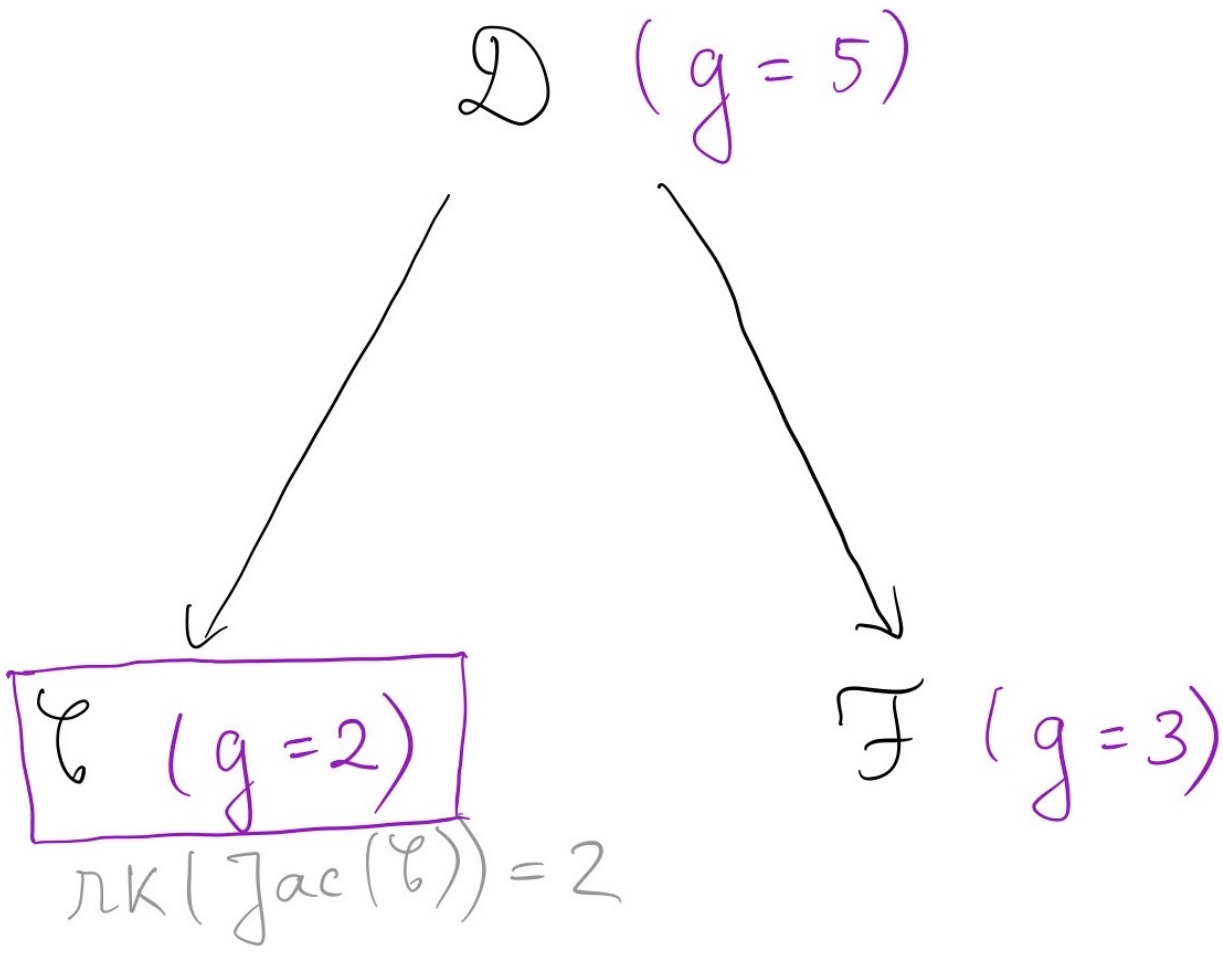
The Fermat quartic

$$x^4 + y^4 = 17$$

has exactly 8 *rational solutions:*

$$\{(\pm 1, \pm 2), (\pm 2, \pm 1)\}.$$

Covering Technique



Understanding the genus 2 curve

- We write the curve in the affine form

$$\boxed{\mathcal{C} : y^2 = F(x) = (9x^2 - 28x + 18)(x^2 + 12x + 2)(x^2 - 2)}.$$

- Proving that $\mathcal{F}(\mathbb{Q}) = \{(\pm 2, \pm 1), (\pm 1, \pm 2)\}$ boils down to showing that $\boxed{\mathcal{C}(\mathbb{Q}) = \{\infty^+, \infty^-\}}$.
- The rest of the proof relies heavily on the fact that $D_1 = [\infty^+ - \infty^-]$ and D_2 are two independent points of infinite order on $\mathrm{Jac}(\mathcal{C})$, where D_2 has Mumford representation $(5x^2 - 18x + 17, 3(-603x + 1187)/50)$.

What about $x^4 + y^4 = 97$?

A natural question to ask after dealing with the $c = 17$ case is whether the same techniques can be employed to solve other values of c which produce the same type of Jacobian. From the point of view of ranks, $c = 17, 97, 257$ look the same, and all of them can be written in the form $c = a^4 + b^4$, with $a, b \in \mathbb{Z}$.

What about $x^4 + y^4 = 97$?

- When we try to replicate the proof from $c = 17$, we again get a genus 2 curve whose Jacobian has rank 2, namely:

$$\mathcal{C} : y^2 = f(x) = (25x^2 - 76x + 50)(x^2 + 28x + 2)(x^2 - 2).$$

- We again would like to find two linearly independent infinite order points on the Jacobian.
- We can again make use of $[\infty^+ - \infty^-]$, but a search in a box could not yield a second point, making us notice that an increase in the value of c may make the cover $\mathcal{C}$ less tractable.

Finding a Point of Infinite Order via Richelot Isogenies

- $J(\mathcal{C}) \sim J(E_1) \times J(E_2)$, given by $\psi_1 : \mathcal{C} \to E_1$ and $\psi_2 : \mathcal{C} \to E_2$
- The elliptic curve E_1 is defined over the quadratic field $L = \mathbb{Q}(\sqrt{97})$ and $E_1(L)$ has rank 2. With the help of Magma, we are able to find two independent points of infinite order in $E_1(L)$.
- Taking the preimage of a point in $E_1(L)$ under the covering map ψ_1 we get a point on $\mathcal{C}$ defined over a quartic extension of $\mathbb{Q}$. Using two such points we can construct a point on $J(\mathcal{C})$ defined over a quadratic extension, and then with two such conjugate points we can construct a point in $J(\mathcal{C})(\mathbb{Q})$.

Conclusion

The Fermat quartic

$$x^4 + y^4 = 97$$

has exactly 8 rational solutions:

$$\{(\pm 3, \pm 2), (\pm 2, \pm 3)\}.$$

Thank you for your attention!

Explicit arithmetic of superelliptic curves and jacobians

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June 2, 2020

A superelliptic curve is the smooth projective model of $y^n = f(x)$ over a field K such that $\mathrm{char}(K) \nmid n$, $f(x) \in K[x]$ is separable, and $(n, \deg f) = 1$. A superelliptic curve with $n = 2$ is an odd-degree hyperelliptic curve.

Let $\mathcal{C}$ be a superelliptic curve of genus g and let $\mathcal{J}$ be its jacobian. The automorphism $\zeta\colon (x, y) \mapsto (x, \zeta_n y)$ of $\mathcal{C}$ induces an automorphism ζ of $\mathcal{J}$.

I did the following in my thesis.

1. I provided a formula for "division by $1 - \zeta$" for points of $\mathcal{C}$. Given a point P of $\mathcal{C}$, I write down formulas for every effective degree g divisor such that $[(1 - \zeta)D] = [P - \infty]$. When $n = 2$, this is the same as division by 2, and reduces to Zarhin's formulas for division by 2 on odd-degree hyperelliptic curves. As an application, one can divide points of the form $(\alpha, 0)$ on P to describe all elements of $\mathcal{J}[(1 - \zeta)^2]$. When $n = 3$, $\mathcal{J}[(1 - \zeta)^2] = \mathcal{J}[3]$, so we can write down all the 3-torsion of the jacobian.

2. The roots of the L-polynomial of the "superelliptic Catalan curve" $y^p = x^q + 1$ are Jacobi sums $J(\chi_p, \chi_q)$ where χ_p and χ_q are characters of order p and q. I found new congruences for such Jacobi sums and use these congruences to give an explicit description of the fields of definition of $\mathcal{J}[p]$ and $\mathcal{J}[q]$.
3. A torsion point of $\mathcal{C}$ is a geometric point P of $\mathcal{C}$ such that $[P - \infty]$ has finite order in $\mathcal{J}$. I classified torsion points on $y^n = x^d + 1$ when $n, d, g \geq 2$ and $(n, d) = 1$, generalizing earlier work of Grant-Shaulis, who considered the case when $n = 2$ and d is prime. Any point with $x = 0$ or $y = 0$ is automatically a torsion point; call a torsion point *exceptional* if $x \neq 0$ and $y \neq 0$. I show that the exceptional cases only occur when $n + d = 7$, and I determine all exceptional torsion points in these cases.

Separating periods of quartic surfaces

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Periods to algebraic cycles

Periods $=\{$ Integrals of algebraic functions over algebraic domains$\}$

$$\pi = 2\int_{-1}^{1} \sqrt{1-x^2}\,\mathrm{d}x$$

$a, b \in$ Periods, $a = b \iff \exists$ an algebraic cycle in a variety over $\overline{\mathbb{Q}}$

$X = Z(f) \subset \mathbb{P}^3$, $f \in \mathbb{Z}[x,y,z,w]_4$, $\omega_f \in \mathrm{H}^{2,0}(X/\mathbb{Q})$

$$\text{Periods of } X = \left\{ \int_\gamma \omega_f \;\middle|\; \gamma \in \mathrm{H}_2(X,\mathbb{Z}) \right\}.$$

Lefschetz (1,1)-theorem: $\int_\gamma \omega_f = 0 \iff \gamma = [C_1] - [C_2]$

An analogy with algebraic numbers

$\alpha \in \overline{\mathbb{Q}}$:	(min poly., approx.)	degree, height
$\int_\gamma \omega_f$:	(f, γ)	Δ_γ, $\lvert f \rvert$.

There is an effective constant $\varepsilon(\Delta_\gamma, f)$ such that:

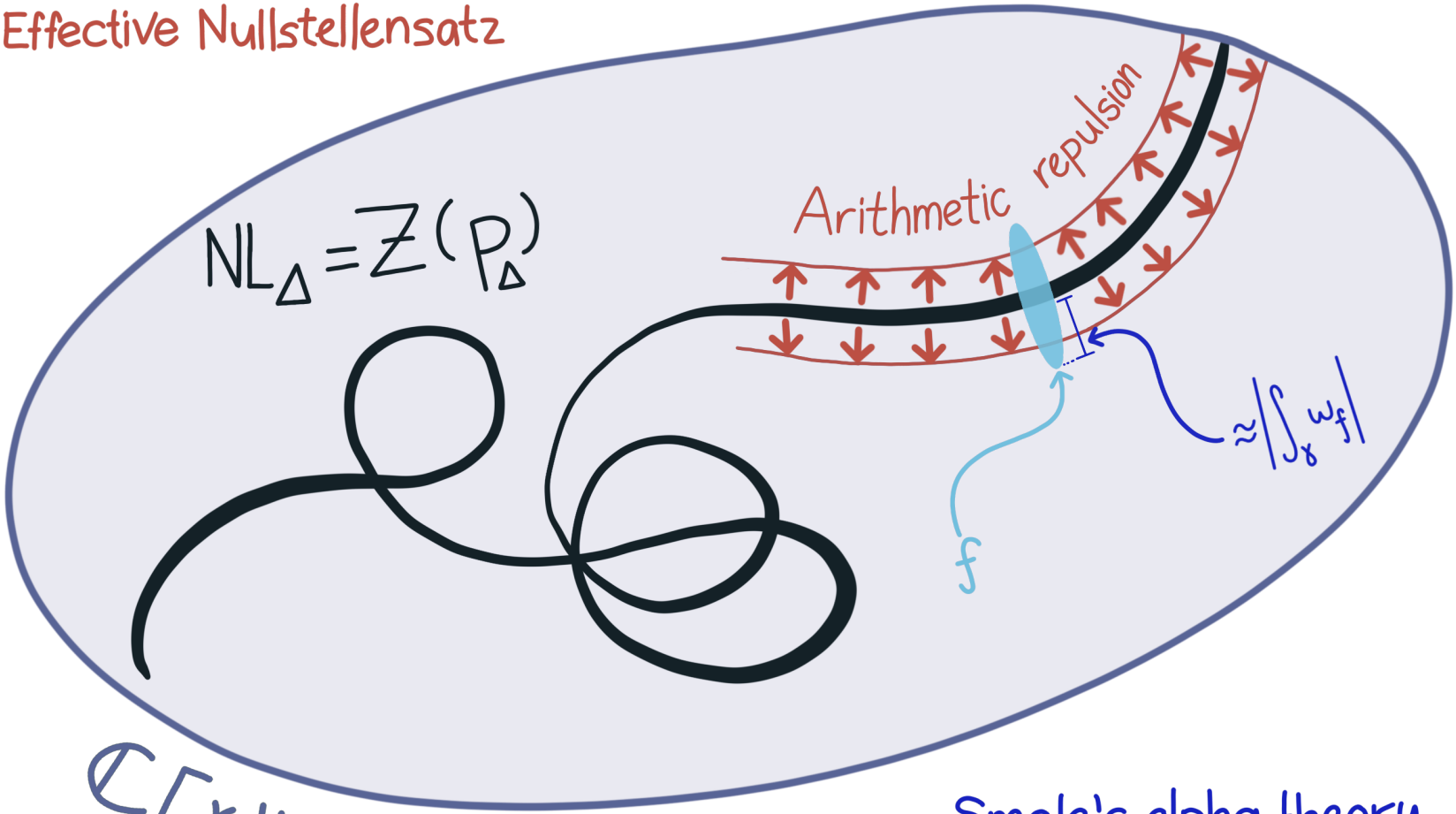
$$\int_\gamma \omega_f = 0 \quad \text{or} \quad \left| \int_\gamma \omega_f \right| > \varepsilon(\Delta_\gamma, f).$$

$$\varepsilon(\Delta, f) = \frac{\left(d_\Delta! |p_\Delta| \left(1 + |f|\right)^{d_\Delta} \right)^{-1}}{4 \| \mathrm{d}\mathcal{P}_f^{-1} \| \left(1 + 6 \frac{\|A\|}{\mathrm{vol}(X_f)} \right)}$$

$\sum_{n \geq 0} \left(2 \uparrow\uparrow (3^n)\right)^{-1}$ is not a ratio of periods of a quartic surface/$\mathbb{Q}$.

Idea

Hilbert schemes
Effective Nullstellensatz



$\mathbb{C}[x,y,z,w]_4 \cong \mathbb{C}^{35}$

Smale's alpha theory
Griffiths–Dwork reduction