Abelian Divisibility Sequences

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Arithmetic of Low-Dimensional Abelian Varietes

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the Fibonacci sequence $1, 1, 2, 3, 5, 8, 13, \ldots$.

An **elliptic divisibility sequence** (**EDS**) is formed from an elliptic curve $E/\mathbb{Q}$ and a non-torsion point $P \in E(\mathbb{Q})$ by writing

$$nP = \left(\frac{A_n(P)}{D_n(P)^2}, \frac{B_n(P)}{D_n(P)^3}\right).$$

The sequence $\big(D_n(P)\big)_{n\ge 1}$ is an EDS.

Divisibility Sequences over Dedekind Domains

More generally, if R is a Dedekind domain, we define an R**-divisibility sequence** to be a sequence of ideals

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In this way we can define an EDS, for example, by factoring the ideal generated by $x(nP)$ in the form

$$x(nP)R = \mathfrak{A}_n(P)\mathfrak{D}_n(P)^{-2}$$

and taking the sequence

$$\big(\mathfrak{D}_n(P)\big)_{n\ge1}.$$

Reformulating EDS

Let E/K be an elliptic curve, let $P \in E(K)$, and let $\mathcal{E}/R$ be a Néron model for E/K. Then the EDS

$$\boxed{\bigl(\mathfrak{D}_n(P)\bigr)_{n\ge 1}}$$

is characterized by noting that for each prime ideal $\mathfrak{p}$ of R, we have*

$$\boxed{\operatorname{ord}_{\mathfrak{p}} \mathfrak{D}_n(P) = \Bigl(\text{largest } k \text{ so that } nP \equiv \mathcal{O} \pmod{\mathfrak{p}^k}\Bigr).}$$

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Or we can simply say that $\mathfrak{D}_n(P)$ is the largest ideal (ordered by divisibility) such that

$$nP \equiv \mathcal{O} \pmod{\mathfrak{D}_n(P)}.$$

Abelian Divisibility Sequences: Type I

In general:

R	a Dedekind domain.
K	the fraction field of R.
A/K	an abelian variety.
$\mathcal{A}/R$	a Néron model for A/K.
$P \in A(K)$	a non-torsion point.

The **abelian divisibility sequence** (**ADS**) for the pair (A, P) is the sequence of ideals $\bigl(\mathfrak{D}_n(P)\bigr)_{n\ge1}$ defined by the property that $\mathfrak{D}_n(P)$ is the largest ideal satisfying

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Alternatively, letting $\pi : \mathcal{A} \to \operatorname{Spec}(R)$, we can define $\mathfrak{D}_n(P)$ via arithmetic intersection theory,

$$\mathfrak{D}_n(P) := \pi_*(nP \cdot \mathcal{O}) = (nP)^*(\mathcal{O}) \in \operatorname{Div}\big(\operatorname{Spec}(R)\big).$$

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Exercise: Prove that $\bigl(\mathfrak{D}_n(P)\bigr)$ is a divisibility sequence.

Growth Rates

A $\mathbb{G}_m$-divisibility sequence $\mathcal{D} = (D_n)$ such as $a^n - b^n$ or the Fibonacci sequence grows exponentially,

$$\lim_{n\to\infty} |D_n|^{1/n} > 1.$$

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Elliptic divisiblity sequences $\mathcal{D} = \bigl(\mathfrak{D}_n(P)\bigr)$ grow even faster,

$$\lim_{n\to\infty} \bigl(\mathrm{N}_{K/\mathbb{Q}}\mathfrak{D}_n(P)\bigr)^{1/n^2} > 1. \qquad (*)$$

Two remarks about elliptic divisibility sequences:

- The limit in $(*)$ is $\hat{H}_E(P)$, i.e.,

$$\mathrm{N}_{K/\mathbb{Q}}\mathfrak{D}_n(P) \approx \hat{H}_E(P)^{n^2} = \hat{H}_E(nP).$$

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The height of nP on an abelian variety grows at a similar rate, but co-dimension considerations suggest that $\mathfrak{D}_n(P)$ might not grow that fast.

Growth Rates of ADS: A Conjecture

Conjecture 1. Let A/K be an abelian variety of dimension ≥ 2, and let $P \in A(K)$ be a point such that $\mathbb{Z}P$ is Zariski dense in A. Then

$$\lim_{n\to\infty} \bigl(\mathrm{N}_{K/\mathbb{Q}}\mathfrak{D}_n(P)\bigr)^{1/n^2} = 1.$$

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Theorem. Conjecture 1 follows from Vojta's conjecture applied to A blown up at $\mathcal{O}$.

A Multiplicative Analogue to Conjecture 1

Here is a $\mathbb{G}_m$ analogue. We replace A by $\mathbb{G}_m^2$ and $P \in A(K)$ with $(a, b) \in \mathbb{G}_m^2(\mathbb{Q})$. The associated divisibility sequence

$$\gcd(a^n - 1, b^n - 1)$$

measures the "arithmetic distance" from $(a, b)^n$ to $(1, 1)$.

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Theorem. (Bugeaud–Corvaja–Zannier 2003) Let $a, b \in \mathbb{Z}$ with $|a| > |b| > 1$. Then

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The proof uses Schmidt's subspace theorem and is surprisingly intricate, even for $a = 3$ and $b = 2$.

Challenge: Give an elementary proof that

$$\gcd(3^n - 1, 2^n - 1)^{1/n} \longrightarrow 1.$$

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Conjecture 2. Let A/K be an abelian variety of dimension ≥ 2, and let $P \in A(K)$ be a point such that $\mathbb{Z}P$ is Zariski dense in A. Then there is a constant $C = C(A/K, P)$ with the property that

$$(*) \quad \left|\mathrm{N}_{K/\mathbb{Q}} \mathfrak{D}_n(P)\right| \leq C \quad \text{for infinitely many } n \geq 1.$$

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Bolder Conjecture: The set of *primes* n such that the inequality $(*)$ holds is a set of positive density.

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Even Bolder ~~Conjecture~~ Question: The set of *primes* n such that the inequality $(*)$ holds is a set of density 1.

A Multiplicative Analogue to Conjecture 2

It is surprising to me that this conjecture wasn't formulated until quite recently.

Conjecture. (Ailon–Rudnick 2004) Let $a, b \in \mathbb{Z}$ with $|a| > |b| > 1$. Then there are infinitely many values of $n \geq 1$ such that

$$\gcd(a^n - 1, b^n - 1) = \gcd(a - 1, b - 1).$$

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Theorem. (Ailon–Rudnick 2004) Let $a(T), b(T) \in \mathbb{C}[T]$ be multiplicatively independent modulo $\mathbb{C}^*$. Then there is a $c(T) \in \mathbb{C}[t]$ such that

$$\gcd\bigl(a(T)^n - 1, b(T)^n - 1\bigr) \mid c(T) \quad \text{for all } n \geq 1.$$

Experiments

I've gathered a fair amount of data for the $\mathbb{G}_m^2$ conjecture, i.e., Ailon–Rudnick's conjecture for

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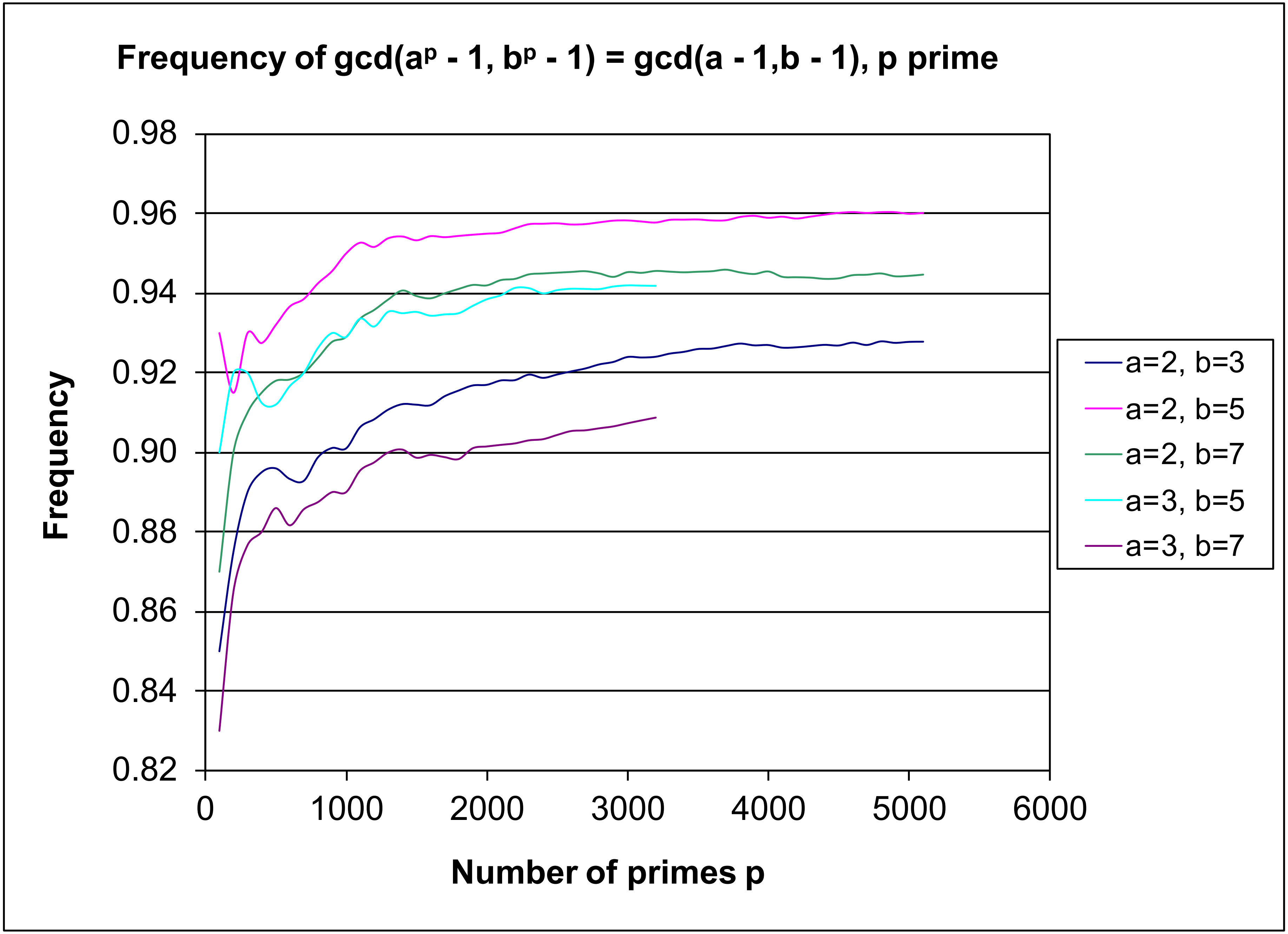
It would be interesting to do some experiments using simple abelian surfaces.



Question: Does the frequency go to 100%?

Fast Growth Versus Slow Growth

As we have seen the growth rate of the ADS associated to an abelian variety A is

$$\boxed{\begin{array}{ll} \textbf{Fast} & \text{if } \dim(A) = 1; \\ \textbf{Slow} & \text{if } \dim(A) \ge 2 \text{ (conjecturally).} \end{array}}$$

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There are a number of reasons why fast-growing divisibility sequences are useful, including:

- Existence of primitive prime divisors (defined later);
- Applications to logic, Hilbert's 10th problem;
- Applications to cryptography based on discrete logarithms and/or pairings.

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How can we get fast-growing, geometrically defined, abelian divisibility sequences in high dimension?

Abelian Divisibility Sequences: Type II

Let $\pi : \mathcal{A} \to \operatorname{Spec}(R)$ be a Néron model for A/K. For a point $P \in A(K)$ on the generic fiber, let $\overline{P} \subset \mathcal{A}$ be the closure of P. We defined the Type I ADS for (A, P) to be the sequence of ideals

$$\mathfrak{D}_n(P) := \pi_*\bigl(\overline{[n]P} \cdot \overline{\mathcal{O}}\bigr) = \pi_*\bigl(\overline{P} \cdot [n]^*\overline{\mathcal{O}}\bigr).$$

This is small when $\dim(A) \geq 2$ because

$$\dim(\mathcal{A}) \geq 3, \quad \text{while} \quad \dim \overline{P} = \dim \overline{\mathcal{O}} = 1.$$

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And in order to get a *divisibility sequence*, we need to replace P, not $\mathcal{O}$.

Abelian Divisibility Sequences: Type II

Definition: Let $X \subset A_K$ be an irreducible codimension 1 subvariety defined over K, and let $\overline{X}$ be its closure in $\mathcal{A}$. The **abelian divisibility sequence** for the pair (A, X) is the sequence of ideals*

$$\mathfrak{D}_n(X) := \pi_*\bigl(\overline{X} \cdot [n]^*\overline{\mathcal{O}}\bigr).$$

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Conjecture. If A is simple, or more generally if X contains no translates of abelian subvarieties, then $\mathfrak{D}_n(X)$ is fast-growing:

$$\liminf_{n\to\infty} \left|\mathrm{N}_{K/\mathbb{Q}}\,\mathfrak{D}_n(X)\right|^{1/n^{2\dim A}} > 1.$$

Tori Divisibility Sequences: Type II

The analogous problem for $\mathbb{G}_m^N$ is solved.

Theorem. (Habegger, Dimitrov, 2016) Let $f \in R[T_1^{\pm 1}, \dots, T_N^{\pm 1}]$ be a Laurent polynomial, and let $X_f \subset \mathbb{G}_m^N$ be the associated divisor. Then

$$\lim_{n\to\infty} \left| \mathrm{N}_{K/\mathbb{Q}} \mathfrak{D}_n(X_f) \right|^{1/n^N} = \mathrm{MahlerMeasure}(f).$$

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Remark. The theorem is false if we allow f to have $\mathbb{C}$ coefficients. Even for $N = 1$, the theorem requires some sort of estimate coming from linear-forms-in-logarithms, because we need to know that algebraic numbers cannot be too closely approximated by roots of unity.

Primitive Prime Divisors and Zsigmondy Sets

Question: Which terms contain a "new" prime divisor?

Definition: Let $\mathcal{D} := (\mathfrak{D}_n)_{n \geq 1}$ be a sequence of ideals. A **primitive prime divisor of $\mathfrak{D}_n$** is a prime ideal $\mathfrak{p}$ satisfying

$$\mathfrak{p} \mid \mathfrak{D}_n \quad \text{and} \quad \mathfrak{p} \nmid \mathfrak{D}_m \quad \text{for all } m < n.$$

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Some Sample Results:

- Bang/Zsigmondy (1886/92): $\mathcal{Z}(a^n - b^n) \subseteq \{1, 2, 6\}$.
- Carmichael (1913): $\mathcal{Z}(\text{Fibonacci sequence}) = \{1, 2, 6, 12\}$.
- Bilu–Hanrot–Voutier (2001):
  $$\mathcal{Z}(\text{Lucas/Lehmer sequence}) \subset \{1, 2, \dots, 30\}.$$
- JS (1988): $\mathcal{Z}(\text{elliptic divisibility sequence})$ is finite.

Primitive Prime Divisors in Abelian Divisibility Sequences

There are two ingredients that go into proofs that the Zsigmondy set of a sequence $\mathcal{D} = (\mathfrak{D}_n)_{n\ge 1}$ is finite:

- The sequence grows rapidly in norm, e.g.,
$$\log \mathrm{N}_{K/\mathbb{Q}} \mathfrak{D}_n \gg n^{\delta} \quad \text{for some } \delta > 1.$$
- The $\mathfrak{p}$-divisiblity does not grow too rapidly, e.g., let r be the smallest index with $\mathfrak{p} \mid \mathfrak{D}_r$, then
$$\operatorname{ord}_{\mathfrak{p}} \mathfrak{D}_{nr} = \operatorname{ord}_{\mathfrak{p}} \mathfrak{D}_r + O\big(\operatorname{ord}_{\mathfrak{p}}(n)\big).$$

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$$\log \mathrm{N}_{K/\mathbb{Q}} \mathfrak{D}_n \gg n^\delta \quad \text{for some } \delta > 1.$$

- The $\mathfrak{p}$-divisiblity does not grow too rapidly, e.g., let r be the smallest index with $\mathfrak{p} \mid \mathfrak{D}_r$, then

$$\operatorname{ord}_{\mathfrak{p}} \mathfrak{D}_{nr} = \operatorname{ord}_{\mathfrak{p}} \mathfrak{D}_r + O\big(\operatorname{ord}_{\mathfrak{p}}(n)\big).$$

For ADS of Type I, the Ailon–Rudnick conjecture implies that $\mathcal{Z}(\mathcal{D})$ is infinite.

Primitive Prime Divisors in Abelian Divisibility Sequences

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For ADS of Type II, we conjecturally have rapid norm growth, but $\mathfrak{p}$-divisibility when $\dim \ge 2$ is much less regular than for $\dim = 1$. Again, I've done experiments and have weak partial results for $\mathbb{G}_m^2$, but it would be very interesting to gather data for abelian surfaces.

Primes in Divisibility Sequences

Let $\mathcal{D} = (D_n)_{n\geq 1}$ be a divisibility sequence in $\mathbb{Z}$.

Natual Question: Are there infinitely many primes in the sequence $|D_n/D_1|$?

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Three Guesses:

- For a $\mathbb{G}_m$-sequence, which typically satisfies

$$\log |D_n| \gg\ll n,$$

  we expect D_p/D_1 to be prime for infinitely many p.
  Example: Mersenne sequence $D_n = 2^n - 1$.

- For an EDS , which typically satisfies

$$\log |D_n| \gg\ll n^2,$$

  we expect D_p/D_1 to be prime for only finitely many p.

- Ditto for higher dimensional Type II ADS with

$$\log |D_n| \gg\ll n^d.$$

I want to thank the organizers,

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Bjorn, Drew, and John,

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Abelian Divisibility Sequences

Joseph H. Silverman

Brown University

Arithmetic of Low-Dimensional Abelian Varietes

ICERM, June 3–7 2019