Abelian varieties with everywhere good reduction over certain real quadratic fields of small discriminant

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Arithmetic of Low-Dimensional Abelian Varieties — ICERM
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Motivation

Theorem (Abrashkin-Fontaine)

There are no abelian varieties defined over $\mathbb{Q}$ with everywhere good reduction.

Applications:

1. Serre conjecture (over $\mathbb{Q}$): The non-existence of abelian varieties with everywhere good reduction over $\mathbb{Q}$ is the opening gambit in the proof of Khare-Wintenberger.
2. Unramified motives: Theorem highlights the importance of motives with little ramification in number theory and arithmetic geometry.

   For example, a better understanding of such motives would leads to new methods for solving Diophantine problems.

Motivation

Main idea of proof:

Let $A/\mathbb{Z}$ be such an abelian scheme, and $A[p]$ the finite group scheme of p-torsion points for a given prime p.

Odlyzko bounds imply that, for certain small primes p (e.g. $p = 3$), the field $L = \mathbb{Q}(A[p])$ (generated by the points of $A[p]$) has a very small root discriminant, and that $L \subseteq \mathbb{Q}(\zeta_p)$.

1. The only simple p-group schemes over $\mathbb{Z}$ are $\mathbb{Z}/p\mathbb{Z}$ and μ_p;
2. $\mathrm{Ext}^1(\mathbb{Z}/p\mathbb{Z}, \mu_p) = \mathrm{Ext}^1(\mu_p, \mathbb{Z}_p) = 0$;
3. Faltings: The p-divisible group attached to A is
$$G \simeq (\mathbb{Q}_p/\mathbb{Z}_p)^g \oplus (\mu_{p^\infty})^g, \ \dim A = g;$$
4. For all integers $n \geq 1$, A has a torsion point of order p^n;
5. This contradicts the boundedness of torsion.

Motivation

Further work:

Over $\mathbb{Q}$: Brumer-Kramer, Calegari and Schoof:

There is no non-zero semistable abelian variety $A/\mathbb{Q}$ with good reduction outside N, for any $N \in \{1, 2, 3, 5, 6, 7, 10, 13\}$.

Over other number fields:

Fontaine's work also showed that this is true for $F = \mathbb{Q}(\sqrt{5})$, $\mathbb{Q}(\sqrt{-1})$ and $\mathbb{Q}(\sqrt{-3})$.

Schoof extended those results to cyclotomic fields. In some unpublished work, he also proved some classification results for real quadratic fields of discriminant ≤ 37.

In all those cases, the Odlyzko bounds imply that the field $L/\mathbb{Q}$ is solvable. So, one can use class field theory to determine L.

Motivation

Going beyond discriminant 37 seemed very challenging. Indeed, even under GRH, Odlyzko bounds grow very fast, and it quickly appears that there are many non-solvable L.

Goal:

In this talk, we will explain some new methods which allows us to deal with larger Odlyzko bounds. This allows us to classify abelian varieties with everywhere good reduction over several real quadratic fields that were beyond reach before.

Motivation

Main result

Theorem (D.)

Let $F = \mathbb{Q}(\sqrt{53}), \mathbb{Q}(\sqrt{61})$ or $\mathbb{Q}(\sqrt{73})$. Then, we have the following:

1. *There exists a simple abelian surface A of GL_2-type over F with everywhere good reduction;*
2. *Under GRH, every abelian variety B over F with everywhere good reduction is isogenous to A^g for some integer $g \geq 1$. In particular, there is no abelian variety of odd dimension over F with everywhere good reduction.*

Motivation

<u>**Main result: Case $F = \mathbb{Q}(\sqrt{53})$**</u>

Let $F = \mathbb{Q}(\sqrt{53})$, and $\mathcal{O}_F = \mathbb{Z}[w]$ the ring of integers of F, where $w = \frac{1+\sqrt{53}}{2}$. Let $C : y^2 + Q(x)y = P(x)$ be the curve over F given by

$$\begin{aligned} P &:= -4x^6 + (w-17)x^5 + (12w-27)x^4 + (5w-122)x^3 \\ &\quad + (45w-25)x^2 + (-9w-137)x + 14w + 9, \\ Q &:= wx^3 + wx^2 + w + 1. \end{aligned}$$

Let $A = \mathrm{Jac}(C)$ be the Jacobian of C. The curve C has discriminant $\Delta_C = -\epsilon^7$, where ϵ is the fundamental unit of F. Thus, the surface A has trivial conductor and RM by $\mathbb{Z}[\sqrt{2}]$.

So, up to isogeny, A is the unique simple abelian variety over $\mathbb{Q}(\sqrt{53})$ with everywhere good reduction.

Motivation

Main result: Case $F = \mathbb{Q}(\sqrt{61})$

Let $S_2(61, (\frac{61}{\cdot}))$ be the space of cusp forms of weight 2, level 61 and quadratic character $(\frac{61}{\cdot})$. This is a 4-dimensional space, which consists of a single Hecke orbit.

Let f be a newform in this Hecke orbit. Then, f has coefficients in the CM quartic field $K_f = \mathbb{Q}(\sqrt{3}, \alpha)$, where $\alpha^2 = -4 + \sqrt{3}$.

By the Eichler-Shimura construction, there is an abelian fourfold B_f with RM by K_f associated to (the Hecke orbit of) f.

Fact:
There is an abelian surface A over F such that

$$B_f \times_{\mathbb{Q}} F \sim A \times A^{\sigma}, \text{ where } \mathrm{Gal}(F/\mathbb{Q}) = \langle \sigma \rangle.$$

The surface A has RM by $\mathbb{Q}(\sqrt{3})$. Up to isogeny, it is the unique simple abelian variety over $\mathbb{Q}(\sqrt{61})$ with everywhere good reduction.

Motivation

Main result: Case $F = \mathbb{Q}(\sqrt{73})$

Let $F = \mathbb{Q}(\sqrt{73})$, and $\mathcal{O}_F = \mathbb{Z}[w]$ be the ring of integers of F, where $w = \frac{1+\sqrt{73}}{2}$. Let $C : y^2 + Q(x)y = P(x)$ be the curve over F given by

$$\begin{aligned} P &:= (w-5)x^6 + (3w-14)x^5 + (3w-19)x^4 + (4w-3)x^3 \\ &\quad + (-3w-16)x^2 + (3w+11)x + (-w-4); \\ Q &:= x^3 + x + 1. \end{aligned}$$

Let $A = \mathrm{Jac}(C)$ be the Jacobian of C. The discriminant of the curve C is $\Delta_C = \epsilon^2$, where ϵ is the fundamental unit of F. Thus, the surface A has trivial conductor. It also has RM by $\mathbb{Z}[\frac{1+\sqrt{5}}{2}]$.

So, up to isogeny, A is the unique simple abelian variety over $\mathbb{Q}(\sqrt{73})$ with everywhere good reduction.

Motivation

Main result: Strategy of proof

The main steps in the proof of Theorem 2 are as follows:

1. Determine all splitting fields $L = F(\underline{M})$, where $\underline{M}$ is a finite flat 2-group scheme over $\mathcal{O}_F$;
2. Classify all simple finite flat 2-group schemes over $\mathcal{O}_F$;
3. Determine all extensions of finite flat 2-group schemes over $\mathcal{O}_F$;
4. Classify all abelian varieties with everywhere good reduction over F.

Steps (2), (3) and (4) are quite hard in general. BUT, more importantly, they all depend on Step (1) which can be even more challenging.

In this talk, I am going to focus mainly on Step (1).

Some facts about group extensions

We say that G is an extension of Q by N if there is an exact sequence

$$1 \to N \to G \xrightarrow{\varphi} Q \to 1.$$

So, we can identify N with a normal subgroup of G.

Let G be an extension of Q by N, and G' a subgroup of G. Then, G' is an extension of Q' by N' where $N' = G' \cap N$ and $Q' = \varphi(G')$.

$$\begin{tikzcd} 1 \arrow[r] & N \arrow[r] & G \arrow[r, "\varphi"] & Q \arrow[r] & 1 \\ 1 \arrow[r] & N' \arrow[r] \arrow[u, hook] & G' \arrow[r, "\varphi"] \arrow[u, hook] & Q' \arrow[r] \arrow[u, hook] & 1 \end{tikzcd}$$

In that case, we have $[G : G'] = [N : N'][Q : Q']$.

Some facts about group extensions

Theorem

Let N and Q be groups, and $\mathrm{Ext}^1(Q, N)$ the set isomorphism classes of extensions of Q by N. Then we have an exact sequence

$$1 \to H^2(Q, \mathrm{Cent}(N)) \to \mathrm{Ext}^1(Q, N) \to \mathrm{Hom}(Q, \mathrm{Out}(N)).$$

In particular, when $\mathrm{Cent}(N)$ is trivial, we have

$$\mathrm{Ext}^1(Q, N) \simeq \mathrm{Hom}(Q, \mathrm{Out}(N)).$$

Some facts about group extensions

Proposition

Let q be a prime power, and $N = \mathrm{PSL}_2(\mathbb{F}_q)$ or $\mathrm{SL}_2(\mathbb{F}_q)$. Let G an extension of Q by N. Then, G has a subgroup of index $q+1$.

Example:

The group $N = \mathrm{PSL}_2(\mathbb{F}_7)$ has a subgroup of index 7. By Theorem 3, there are three extensions of D_4 by N. BUT only the trivial extension has a subgroup of index 7.

Lemma

Let N be a non-solvable group of order 60, 120, 180 or 240, and G an extension of D_4 by N. Then G has a subgroup of index 5.

The Fontaine bound

Theorem (Fontaine)

Let p a prime, $K/\mathbb{Q}_p$ a finite extension, and $\mathcal{O}_K$ the ring of integers of K. Let $n \geq 1$ be an integer and $\underline{M}$ a finite group scheme over $\mathcal{O}_K$ killed by n. Let e be the absolute ramification index of K, and $G^{(u)}$ $(u \geq -1)$, the higher ramification groups.
If $u \geq e\left(n + \frac{1}{p-1}\right)$, then $G^{(u)}$ acts trivially on $\underline{M}(\overline{K})$.

Another way of saying this is that that the Galois action of $\mathrm{Gal}(\overline{K}/K)$ on $\underline{M}(\overline{K})$ is très peu ramifiée.

The Fontaine bound

Theorem (Fontaine)

Let p be a prime, F a number field and A an abelian variety defined over F. Assume that A has everywhere good reduction. Let $L = F(A[p])$ (the field generated by the p-torsion points). Then, we have

$$\delta_L < \delta_F p^{1+\frac{1}{p-1}},$$

where δ_F and δ_L are the root discriminants of F and L.

Number fields arising from weight 1 modular forms

Theorem

There is no number field $K/\mathbb{Q}$ which is très peu ramifié at 2 and tamely ramified at 61, with Galois group $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$ and root discriminant $\delta_K < 4 \cdot 61^{1/2} = 31.2409\ldots$.

Sketch of proof:

1. K comes from an odd representation $\tilde{\rho}:\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{PSL}_2(\mathbb{F}_7)$;
2. We can lift $\tilde{\rho}$ to a $\bar{\rho}:\ \mathrm{Gal}(\overline{\mathbb{Q}}/\mathbb{Q}) \to \mathrm{GL}_2(\overline{\mathbb{F}}_7)$, which is still très peu ramifié at 2 and tamely ramified at 61;
3. The representation $\bar{\rho}$ comes from a non-liftable mod 7 modular form of weight one, and level dividing $8 \cdot 61$.
4. We show that no such form exists.

Number fields arising from weight 1 modular forms

Example

Let K be the splitting field of the polynomial

$$h = x^7 - 2x^6 - 4x^5 + 6x^4 + 8x^3 - 22x^2 + 16x - 2.$$

We have $\mathrm{Gal}(K/\mathbb{Q}) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$*, and K is ramified at* 2 *and* 61 *only, and* $\delta_K = 2^{6/7} \cdot 61^{3/4} = 39.5387\ldots$.

In this case, the field K comes from a mod 7 modular form of weight one and level $2 \cdot 61$ *and Dirichlet character of order* 4.

So, in some sense, the root discriminant bound $4 \cdot 61^{1/2} = 31.2409\ldots$ *is quite sharp.*

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{61})$

Lemma

Under GRH, there is no non-solvable Galois extension $L/\mathbb{Q}$ unramified outside of 2 and 61 such that $E := F(\sqrt{-1}, \sqrt{\epsilon})$ is contained in L, and root discriminant $\delta_L < 4 \cdot 61^{1/2} = 31.2409\ldots$.

Proof:

Let $L/\mathbb{Q}$ be such a Galois extension. Under GRH, the Odlyzko bounds yield that $[L : \mathbb{Q}] < 2400$. So we have an inclusion of fields

$$\mathbb{Q} \subset_8 E \subseteq_{\leq 299} L,$$

with $N = \mathrm{Gal}(L/E)$ and $D_4 = \mathrm{Gal}(E/\mathbb{Q})$ and

$$1 \to N \to G \to D_4 \to 1.$$

1. $\mathrm{Gal}(E/\mathbb{Q})$ is solvable implies that N is non-solvable with $|N| \leq 299$;
2. By Feit-Thompson, N is a group of order $60, 120, 168, 180$ or 240.

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{61})$

Case 1: N has a subgroup of index 5.

1. N has order $60, 120, 180$ or 240. By Lemma 5, G has a subgroup H of index 5.
2. The field $K' := L^H$ has degree 5, and is unramified outside 2 and 61.
3. By the Jones-Roberts' tables, there is a unique number field K' of degree 5 unramified outside 2 and 61, whose normal closure has root discriminant $< 31.2409...$. It is given by the polynomial $x^5 - x^4 - 5x^3 + 13x^2 + 10x + 2$.
4. Let L' be the normal closure of the compositum of E and K'. By direct calculations, we determined that $[L' : \mathbb{Q}] = 80$, thus L' is solvable. Let $G' = \mathrm{Gal}(L'/\mathbb{Q})$, and $N' \triangleleft G$ such that

$$1 \to N' \to G \to G' \to 1.$$

Then, again from the Odlyzko bounds, we see that $|N'| < 30$. Thus N', and hence G, is solvable. This contradiction shows that N cannot have a subgroup of index 5.

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{61})$

Case 1: N has a subgroup of index 8.

1. In this case, $N = \mathrm{PSL}_2(\mathbb{F}_7)$, the unique simple group of order 168.
2. There are three such extensions of D_4 by N. By Proposition 4, each of them has a subgroup H of index 8.
3. The fixed field of H is a number field K' of degree 8. Its normal closure L' is non-solvable and linearly disjoint from E; and we have $\delta_{L'} < 31.2409\ldots$. It follows that $G' = \mathrm{Gal}(L'/\mathbb{Q}) \simeq \mathrm{PSL}_2(\mathbb{F}_7)$.
4. In that case, L' arises from a non-liftable weight one modular form with coefficients in $\overline{\mathbb{F}}_7$ whose level divides $8 \cdot 61$. However, there are no such fields by Theorem 8.

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{61})$

Proposition

Let K be the Hilbert class field of $F(\sqrt{-1}) = \mathbb{Q}(\sqrt{61}, \sqrt{-1})$, and L the splitting field of the the polynomial
$h = x^{12} + 4x^9 + 15x^8 + 4x^7 - 3x^6 + 30x^5 + 49x^4 + 16x^3 - 18x^2 - 14x - 3$.
Then, we have the following:

1. (i) *L contains the field $E := F(\sqrt{-1}, \sqrt{\epsilon})$.*
2. (ii) *L is a 2-extension of K with root discriminant* $\delta_L = 2^{3/2} \cdot 61^{1/2} = 22.0907...$;
3. (iii) $\mathrm{Gal}(L/F) = \mathbb{Z}/2\mathbb{Z} \times S_4 = \mathrm{SL}_2(\mathbb{F}_2[e])$, *with* $e^2 = 0$;
4. (iv) *L/K is the unique Galois extension unramified outside 2 and 61 such that* $\delta_L < 4 \cdot 61^{1/2} = 31.2409...$.

Proof.

The proof uses class field theory. □

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{61})$

Theorem

Let $\underline{M}$ be a finite flat 2-group scheme over $\mathbb{Z}[\frac{1+\sqrt{61}}{2}]$, and $L' = F(\underline{M})$ the field generated by its $\overline{\mathbb{Q}}$-points. Then L' is a subfield of L.

Proof.

This combines Theorem 7 and Proposition 11. $\square$

Finite flat 2-group schemes over $\mathbb{Z}[\frac{1+\sqrt{61}}{2}]$

Theorem

The simple finite flat 2-group schemes over $\mathbb{Z}[\frac{1+\sqrt{61}}{2}]$ are $\mathbb{Z}/2\mathbb{Z}$, μ_2 and $A[\pi]$, where π is a generator of the prime above 2 in $\mathbb{Q}(\sqrt{3})$.

Proof.

Recall that $\mathrm{Gal}(L/F) = \mathrm{SL}_2(\mathbb{F}_2[e])$, and that there is an exact sequence

$$1 \to (\mathbb{Z}/2\mathbb{Z})^3 \to \mathrm{Gal}(L/F) \to S_3 \to 1.$$

If M is an irreducible $\mathbb{F}_2[\mathrm{Gal}(L/F)]$-module, then the action must factor through the Galois group of the fixed field of $(\mathbb{Z}/2\mathbb{Z})^3$. So, M must be an irreducible $\mathbb{F}_2[S_3]$-module.
The irreducible $\mathbb{F}_2[S_3]$-modules have dimensions 1 or 2. One shows that $\mathbb{Z}/2\mathbb{Z}$, μ_2 and $A[\pi]$ are the only simple finite flat 2-group schemes with those Galois modules. $\square$

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{73})$

Theorem

Under GRH, there is a unique non-solvable Galois number field $L/\mathbb{Q}$ unramified outside of 2 and 73 such that $E := F(\sqrt{-1}, \sqrt{\epsilon})$ is contained in L, and the root discriminant $\delta_L < 4 \cdot 73^{1/2} = 34.1760\ldots$. The field L is a 2-extension of the splitting field K of the polynomial

$$h := x^{12} + 4x^{11} - 3x^{10} - 20x^9 + 12x^8 + 30x^7 - 77x^6 + 2x^5 + 210x^4 + 40x^3 - 108x^2 - 56x - 8.$$

Corollary

Let $\underline{M}$ be a finite flat 2-group scheme over $\mathbb{Z}[\frac{1+\sqrt{73}}{2}]$, and $L' = F(\underline{M})$ the field generated by its $\overline{\mathbb{Q}}$-points. Then L' is a subfield of L.

The field of 2-torsion for $F = \mathbb{Q}(\sqrt{73})$

Sketch of proof:

To bound the degree of L, we must consider all extensions of D_4 by N:

1.
   (i) $A_5 \simeq \mathrm{PSL}_2(\mathbb{F}_5)$, $\mathrm{SL}_2(\mathbb{F}_5)$;
   (ii) The two non-trivial extensions of A_5 by $(\mathbb{Z}/2\mathbb{Z})^4$;
2.
   (i) $\mathrm{PSL}_2(\mathbb{F}_7)$, $\mathrm{SL}_2(\mathbb{F}_7)$, $\mathrm{PSL}_2(\mathbb{F}_9)$, $\mathrm{SL}_2(\mathbb{F}_8)$, $\mathrm{PSL}_2(\mathbb{F}_{11})$, $\mathrm{SL}_2(\mathbb{F}_9)$, $\mathrm{PSL}_2(\mathbb{F}_{13})$;
   (ii) The non-trivial extension of $\mathrm{PSL}_2(\mathbb{F}_9)$ by $\mathbb{Z}/3\mathbb{Z}$.

In Case 2, this again requires an extensive calculation of non-liftable weight one forms whose levels divide $8 \cdot 73$.

Concluding remarks

Until now, abelian varieties with everywhere good reduction have been classified only for real quadratic fields of discriminant ≤ 37. This is when GRH Odlyzko bounds guarantee that the field of 2-torsion L is solvable.

The field $F = \mathbb{Q}(\sqrt{73})$ is the first example where the coordinates of the 2-torsion points of the abelian variety A generate a non-solvable extension $L/\mathbb{Q}$. In this case, we have $\mathrm{Gal}(L/F) \simeq A_5$, the smallest simple group.

We believe that our method will extend to yield a complete classification results for all real quadratic fields of discriminants ≤ 100.

D	37	73	97
δ_L	24.3310...	34.1760...	39.3954...
$[L : \mathbb{Q}]$	300	8862	1000,000

Table: Odlyzko bounds for real quadratic fields with discriminant < 100

Thank you for your attention!