Decomposing Jacobian Varieties

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Plan of Attack

- Why decompose Jacobian Varieties?
- How to use group actions to decompose them.
- Some results.
- An intermediate interlude.
- Some work in progress.

Motivating Questions

I. For a fixed genus g what is the largest positive integer t such that there is some genus g curve over $\overline{\mathbb{Q}}$, and some elliptic curve E with Jacobian variety $JX \sim E^t \times A$?

$\dim(JX) = g$ so the largest t can be is g.

Motivating Questions

II. In genus 2, Jacobians of curves with nontrivial automorphism group decompose into the product of two elliptic curves. Often those elliptic curves have interesting arithmetic properties.

III. Ekedahl and Serre [1993] find curves in various genera up to 1297 with a curve X of that genus with

$$JX \sim E_1 \times E_2 \times \cdots \times E_g$$

Motivating Questions

IV. Once we have a decomposition, what are the factors? Do they have complex multiplication? Are there fixed factors across families of curves?

V. Equations of curves are generally hard to come by, so can we answer these questions without access to an equation?

Group Actions

The following technique will work over any field, as long as you know the automorphism group of the corresponding curve over that field.

Given a compact Riemann surface X of genus $g \geq 2$, let $G = \mathbf{Aut}(X)$ (or a possible subgroup) and let $X/G = X_G$ be the quotient, of genus h.

Riemann's Existence Theorem

A finite group G acts on a compact Riemann surface X of genus g >1 if and only if there are elements of the group

$$a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r$$

which generate the group, satisfy the following equation,

$$\prod_{i=1}^{h} [a_i, b_i] \prod_{j=1}^{r} c_j = 1_G$$

and so that $m_j = |c_j|$ satisfy the Riemann Hurwitz formula

$$g = 1 + |G|(h-1) + \frac{|G|}{2} \sum_{j=1}^{r} \left(1 - \frac{1}{m_j}\right).$$

Signature: $(h; m_1, \ldots, m_r)$ Generating vector: $(a_1, b_1, \ldots, a_h, b_h, c_1, \ldots, c_r)$

An idempotent f in $\mathbf{End}_{\mathbb{Q}}(JX)$ produces a factor of the Jacobian: $f(JX)$.

But the endomorphism ring is complicated, so we start with $\mathbb{Q}[G]$ and translate using the natural map

$$\eta : \mathbb{Q}[G] \to \mathbf{End}_{\mathbb{Q}}(JX)$$

From a theorem of Wedderburn we know that

$$\mathbb{Q}[G] \cong M_{n_1}(\Delta_1) \times \cdots M_{n_s}(\Delta_s)$$

where the Δ_i are division rings.

$\pi_{i,j}$ is the idempotent of $\mathbb{Q}[G]$ with the zero matrix in every component except the i^{th} component where it has a 1 in the j, j position only. Then

$$1_{\mathbb{Q}[G]} = \sum_{i,j} \pi_{i,j}\,.$$

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This translates by a result of Kani and Rosen [1989] to an isogeny

$$JX \sim \eta(\pi_{1,1})JX \times \eta(\pi_{1,2})JX \times \cdots \times \eta(\pi_{1,n_1})JX \times \cdots \times \eta(\pi_{s,n_s})JX$$

$$JX \sim B_1^{n_1} \times \cdots B_s^{n_s}$$

We call this the group algebra decomposition.

For a special $\mathbb{Q}$-representation V called the Hurwitz representation with character χ_V

$$\dim B_i = \frac{1}{2}\langle \chi_i, \chi_V \rangle$$

where the χ_i are the irreducible $\mathbb{Q}$-characters of G.

V is the representation of G on $H_1(X, \mathbb{Z}) \otimes \mathbb{Q}$.

Definition

Given a branched cover $X \to X_G$ with monodromy $c_1, \ldots, c_r$,

$$\chi_V = 2\chi_0 + 2(h-1)\rho_{\langle 1_G \rangle} + \sum_{i=1}^{r} (\rho_{\langle 1_G \rangle} - \rho_{\langle c_i \rangle})$$

where $\rho_{\langle c_i \rangle}$ is the character of G induced from the trivial character of the subgroup $\langle c_i \rangle$, and χ_0 is the trivial character of G.

$$JX \sim B_1^{n_1} \times \cdots \times B_s^{n_s} \quad \text{where} \quad \dim B_i = \frac{1}{2}\langle \chi_i, \chi_V \rangle$$

To compute the dimensions of the factors of JX, we need

- Automorphism group of X
- Monodromy of the cover $X \to X_G$
- Irreducible $\mathbb{Q}$-characters

Some Results

I. Hyperelliptic Curves
Their automorphism groups are well known [2013].

II. Hurwitz Curves
Curves with the largest possible automorphism groups for a fixed genus, with signature $[0; 2,3,7]$, **e.g.** $PSL(2,q), A_n$.
[2016] and current work with students.

III. Completely Decomposable Jacobians
with Anita Rojas [2017]

Given a genus g curve X, its Jacobian variety JX is called completely decomposable if

$$JX \sim E_1 \times E_2 \times \cdots \times E_g$$

where the E_i are (possibly isogenous) elliptic curves.

Ekedahl and Serre [1993] demonstrate various curves up to genus 1297 with completely decomposable Jacobian varieties. However, there are numerous “gaps” in their data.

Yamauchi [2007] gives lists of integers N so that the Jacobian of the modular curve $X_0(N)$ is completely decomposable. His work adds genus 113, 161, and 205 to the list.

We applied the technique above to many curves up to genus 101, and a few strategically chosen curves up to genus 500, and found 7 new examples for

$$g = \{36,\ 46,\ 81,\ 85,\ 91,\ 193,\ 244\}\ .$$

- All curves up to genus 48 [Breuer, 2000]
- Curves with automorphism group larger than 4(g-1) for genus up to 101 [Conder, 2010]
- Even higher genus using `LowIndexNormalSubgroup(Γ, n)`

An Intermediate Interlude

The rest of the talk is joint work with
Anita Rojas from Universidad de Chile.

Group actions will not tell the whole story.

Example

Consider the curve $Y : y^2 = (x^2 - 4)(x^3 - 3x + a)$.

- **Aut**$(Y) \cong C_2$ so can't decompose by previous technique.
- For $X : y^2 = x(x^6 + ax^3 + 1)$ we find $JX \sim JY \times E_1$
- We can also compute $JX \sim E_1^2 \times E_2$ and so $JY \sim E_1 \times E_2$.

An Example

There is a genus 101 curve with automorphism group (800, 980) $= C_{10}^2 \rtimes C_8$ whose Jacobian decomposes as:

$$JX \sim E \times A \times E^2 \times \underbrace{E^8 \times \cdots \times E^8}_{12}$$

where A is an abelian variety of dimension 2 .

V_i an irreducible $\mathbb{C}$-representation from the i^{th} irreducible $\mathbb{Q}$-character, and m_i the Schur index.

Carocca and Rodriguez [2006]

Given a Galois cover $X \to X_G$ and

$$JX \sim B_1^{\frac{\dim V_1}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s}{m_s}},$$

if H is a subgroup of G then the group algebra decomposition of JX_H is given as

$$JX_H \sim B_1^{\frac{\dim V_1^H}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s^H}{m_s}},$$

where V_i^H is the subspace of V_i fixed by H.

$$JX \sim B_1^{\frac{\dim V_1}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s}{m_s}} \qquad JX_H \sim B_1^{\frac{\dim V_1^H}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s^H}{m_s}}$$

We can compute $\dim V_i^H$ as $\langle V_i, \rho_H \rangle$.

We look for large genus decompositions that need not have all elliptic curves in their decomposition ...

$$JX \sim B_1^{\frac{\dim V_1}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s}{m_s}} \qquad JX_H \sim B_1^{\frac{\dim V_1^H}{m_1}} \times \cdots \times B_s^{\frac{\dim V_s^H}{m_s}}$$

We can compute $\dim V_i^H$ as $\langle V_i, \rho_H \rangle$.

We look for large genus decompositions that need not have all elliptic curves in their decomposition ...

... then we compute the dimensions of the exponents hoping they are 0 for any i which is not an elliptic curve in the decomposition of JX.

An Example (again)

There is a genus 101 curve with automorphism group (800, 980) whose Jacobian decomposes as:

$$JX \sim E \times A \times E^2 \times \underbrace{E^8 \times \cdots \times E^8}_{12}$$

where A is an abelian variety of dimension 2 .

The group has 3 subgroups which produce quotients of genus 51. Using the previous slide, we get:

$$JX_H \sim E \times E^2 \times \underbrace{E^4 \times \cdots \times E^4}_{12}$$

P. and Rojas (2017)

For every integer g in the following list, there is a curve of genus g with completely decomposable Jacobian variety found using a group acting on a curve.

1–29, **30,** 31, **32,** 33, **34–36**, 37, **39,** 40, 41, **42,** 43, **44,** 45, **46**, 47, **48**, 49, 50, **51–52**, 53, **54,** 55, 57, **58,** 61, 62, **63,** 64, 65, **67, 69, 71,** 72–73, **79–81,** 82, **85, 89, 91, 93, 95,** 97, **103, 105–107**, 109, **118,** 121, **125,** 129, **142,** 145, **154,** 161, 163, **193, 199, 211, 213,** 217, **244**, 257, 325, 433

The numbers in pink are new genera, the others are different examples from those Ekedahl and Serre found.

Algorithmifying

The rest of the talk is work in early stages.

Use prior work of Auffarth, Behn, Lange, Rodríguez, Rojas

$A = V_A / L_A$ a complex torus where V_A is a complex vector space of dimension g and L_A is a lattice in V_A.

A polarization on A is a non-degenerate real alternating form E satisfying

$$E(iu, iv) = E(u, v) \quad \text{and} \quad E(L_A \times L_A) \subseteq \mathbb{Z}\,.$$

A polarization is of type $(d_1, d_2, \ldots, d_g)$ if there exists a basis for L_A such that the matrix for E with respect to that basis has the form

$$E = \begin{pmatrix} 0 & D \\ -D & 0 \end{pmatrix}.$$

where D is the diagonal matrix formed from the type: $(d_1, d_2, \ldots, d_g)$.

Such a basis $\gamma_A = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ is called symplectic.

The period matrix for A is the $g \times 2g$ matrix

$$\Pi_A = (D \quad Z)$$

where Z is a $g \times g$ complex symmetric matrix such that $\Im(Z)$ is positive definite.

Goal: Given a principally polarized abelian variety A, its period matrix, and an idempotent f, compute the period matrix of the corresponding factor $f(A)$.

- If the factor is an elliptic curve: the period matrix will be of the form $(1 \;\; \tau)$.

- If the factor is higher dimensional: Auffarth, Lange and Rojas [2017] give criterion on Neron-Severi group to determine additional factorization.

Let $f \in \mathbf{End}_{\mathbb{Q}}(A)$ be an idempotent defining a subvariety $B = f(A) = V_B/L_B$ of dimension h.

Lift uniquely to a map $V_A \to V_A$ and then restrict this map to the lattice:

$$\rho_r : L_A \to L_A \,.$$

But because we have a group action, this induces a map:

$$\rho_r : G \to GL(L_A \otimes \mathbb{Q}) \,.$$

ρ_r is sometimes called the rational representation of the action of G on A. I prefer symplectic representation.

Since the group action respects the polarization, for all $g \in G$,

$$\rho_r(g) \in Sp_{2g}(\mathbb{Z}).$$

This representation is the key to determining the induced polarization and lattice of the factor B.

E.g., the lattice of B is given by the intersection of the $\mathbb{Z}$ basis formed by the columns of matrix $\rho_r(f)$ intersected with L_A.

Similarly, we can compute the induced polarization.

- $\gamma_A = \{\alpha_1, \ldots, \alpha_g, \beta_1, \ldots, \beta_g\}$ a symplectic basis of L_A .
- Since $\Pi_A = (I_g \ Z)$ then the α_i form a basis of V_A .
- Determine a symplectic basis of L_B, say $\gamma_B = \{u_1, \ldots, u_h, v_1, \ldots, v_h\}$.
- The u_i and v_i can be written with respect to γ_A .

- Use Π_A to replace any β_j with α_i.

- We get a $g \times 2h$ matrix whose columns are γ_B written with respect to a $\mathbb{C}$-basis of V_A. Call it $M_B = (C_1 \; C_2)$.

- C_1 and C_2 will then each form a $\mathbb{C}$-basis of V_B written with respect to $\mathbb{C}$-basis of V_A.

- But then $C_1 D^{-1}$ is too! We want $\left\{ \frac{u_1}{d_1}, \ldots, \frac{u_h}{d_h} \right\}$ basis.

- Find change of basis matrix from $C_1 D^{-1}$ to C_2 in Siegel upper half-space. Call it W.

- Then columns of W are coordinates of $\{v_1, \ldots, v_h\}$ with respect to $\left\{ \frac{u_1}{d_1}, \ldots, \frac{u_h}{d_h} \right\}$.

- Means we can form period matrix: $\Pi_B = (D \;\; W)$.

Results

$C_2 \times C_4 = \langle a \rangle \times \langle b \rangle$ acts on a family of non-hyperelliptic curve of genus 3 with signature [0; 2,2,4,4] in two topologically inequivalent ways.

A generating vector for one action is (a, ab^2, b, b). We can factor the Jacobian as $JX \sim E_1 \times E_2 \times E_3$.

$$\Pi_A = \begin{pmatrix} 1 & 0 & 0 & i & -1 & -\frac{1}{2} - \frac{1}{2}i \\ 0 & 1 & 0 & -1 & i & -\frac{1}{2} + \frac{1}{2}i \\ 0 & 0 & 1 & -\frac{1}{2} - \frac{1}{2}i & -\frac{1}{2} + \frac{1}{2}i & x \end{pmatrix}$$

- $M_{B_1} = \begin{pmatrix} i & -1 \\ 1 & i \\ \frac{1}{2} - \frac{1}{2}i & \frac{1}{2} + \frac{1}{2}i \end{pmatrix}$ $E_1 \sim \mathbb{C}/\langle 1, i \rangle$

- $M_{B_2} = \begin{pmatrix} 1 & i \\ i & -1 \\ -\frac{1}{2} + \frac{1}{2}i & -\frac{1}{2} - \frac{1}{2}i \end{pmatrix}$ $E_2 \sim \mathbb{C}/\langle 1, i \rangle$

- $M_{B_3} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \\ 1 & 2x - i \end{pmatrix}$ $E_3 \sim \mathbb{C}/\langle 1, 2x - i \rangle$

$(96,28) = (C_2 \times C_{24}) \rtimes C_2$ acts on a hyperelliptic curve of genus 11 with signature [0; 2,4,24].

The group algebra decomposition gives

$$JX \sim E_1 \times E_2^2 \times E_3^2 \times E_4^2 \times A^2$$

where A is a surface.

Using our method we get that $E_1 \sim E_3 \sim \mathbb{C}/\langle 1\ i \rangle$ and by the Neron Severi method, that A also factors more.

Issues (so far!)

- What is the period matrix of ambient variety?
- Which parameter value(s) correspond to Jacobian varieties?
- Higher dimension gets hard!

The End