Exceptional Vinberg representations and moduli spaces

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An apology: Although there are low-dimensional abelian varieties in this talk, there is not really arithmetic. However, the results *are* related to doing Bhargavology for:

- 3-Selmer groups of genus 2 curves with a Weierstrass point

- 2-Selmer groups of plane quartics with a flex

- 2-Selmer groups of (non-hyperelliptic) curves of genus 4 with $K_C = 6p$ (a 7-dimensional family)*

(The 2-Selmer cases were dealt with differently by Jack Thorne in char. 0; the approach below works in all characteristics)

*In each case, omitting certain bad primes makes the moduli space open in a weighted projective space; e.g., over $\mathbb{Z}[1/15]$, the third moduli space is naturally an open subset in a weighted projective space with degrees $2, 8, 12, 14, 18, 20, 24, 30$.

Bhargava's results on p-Selmer groups for small p rely on various nice representations (e.g., SL_3^3 on $V_3^{(1)} \otimes V_3^{(2)} \otimes V_3^{(3)}$). These are all instances of a more general construction due to Vinberg: Given a Lie group G with an action of μ_l, this induces a $\mathbb{Z}/l\mathbb{Z}$-grading on $\mathfrak{g}$, and an action of G^{μ_l} on $\mathfrak{g}_1$. Over $\mathbb{C}$, these have diagonalization, nice invariants, etc.

A particularly nice source of examples starts with the grading on $\mathfrak{g}$ by height; reducing this grading on E_6 modulo 3 gives the above example. Other examples relate to 2-Selmer groups of hyperelliptic curves, etc.

Today's talk focuses on the following three exceptional cases:

- SL_9/μ_3 acting on $\wedge^3 V_9$ ($\mathfrak{e}_8$ modulo 3)
- SL_8/μ_4 acting on $\wedge^4 V_8$ ($\mathfrak{e}_7$ modulo 2)
- Spin_{16}/μ_2 acting on V_{128} ($\mathfrak{e}_8$ modulo 2)

For the first case, Gruson/Sam/Weyman tell how to use a trivector to construct a torsor over a principally polarized abelian surface: a point in $\mathbb{P}(V_9)$ induces an element of $\wedge^2 V_8$ (modulo scalars), and the rank 4 locus has the same Hilbert series as a ppas embedded by 3Θ. (They also give conjectures for the other two cases)

Focusing on trivectors for the moment: How can we reverse this? I.e., given a genus 2 curve C (with a marked Weierstrass point p^*), what's the corresponding trivector?

One approach: We can actually compute (an affine patch of) $J(C)$ explicitly. First note that the embedding of C in $\mathrm{Proj}(\bigoplus_n \mathcal{L}(np))$ has equation:

$$y^2 + a_1x^2yw + a_3xyw^3 + a_5yw^5 + \\ x^5 + a_2x^4w^2 + a_4x^3w^4 + a_6x^2w^6 + a_8xw^8 + a_{10}w^{10} = 0$$

(This works over $\mathbb{Z}$; over $\mathbb{Z}[1/10]$, we could eliminate a_1, a_2, a_3, a_5 and the remaining coefficients are independent!) Any torsion-free sheaf on such a curve gives a sheaf on this weighted projective space.

*This comes from Θ on $J(C)$

If $\mathcal{L}$ is torsion-free of rank 1 with $H^0(\mathcal{L}) = H^1(\mathcal{L}) = 0$, then it has Hilbert series $t/(1-t)^2$, and its image has a natural presentation

$$0 \to \mathcal{O}(-6) \oplus \mathcal{O}(-7) \to \mathcal{O}(-1) \oplus \mathcal{O}(-2) \to i_*\mathcal{L} \to 0$$

So the complement of the theta divisor in $J(C)$ can be identified with the space of equivalence classes of matrices

$$\begin{pmatrix} b_0y + b_1x^2 + b_3x + b_5 & c_0x^3 + c_1y + c_2x^2 + c_4x + c_6 \\ d_0x^2 + d_1x + d_2 & e_0y + e_1x^2 + e_3x + e_5 \end{pmatrix}$$

with the appropriate determinant.

The group is *not* reductive, so we can't directly do GIT, but we can actually pin down orbit representatives:

$$\begin{pmatrix} y + b_3 x + b_5 & -x^3 + c_2 x^2 + c_4 x + c_6 \\ x^2 + d_2 x + d_4 & y + e_1 x^2 + e_3 x + e_5 \end{pmatrix}$$

So we get $J(C) \setminus \Theta$ as an explicit complete intersection inside this affine space (and simple enough to compute over $\mathbb{Z}$). Note that the *total space* of this family of Jacobians is just an affine space!

Can do something similar for pairs (C, p) with $K_C = 6p$ (so C is genus 4 and uniquely trigonal)

How can we relate this to trivectors? It turns out that there's a general construction of "Weierstrass forms" in Vinberg representations coming from the height grading. (This generalizes an idea of Kostant for ungraded Lie algebras.) More natural to take $\mathfrak{g}_{-1}$. Then the elements of height ≥ -1 and congruent to -1 modulo l form a Borel-invariant subspace, and any element of $\mathfrak{g}_{-1}$ can be put into this form (the corresponding set of flags is closed, so proper!). If the coefficient of a negative simple root vanishes, then the element is not stable, and we can find an S-equivalent vector with a nonzero coefficient. So any element is S-equivalent to a "subtriangular" element s.t. each negative root vector has coefficient 1.

For a fixed flag, this form is unique up to the action of $U \subset B$, and we can mostly use this to eliminate coefficients (ala completing squares, etc.).

For trivectors, we find in this way that every trivector is S-equivalent to one of the form

$$[267]+[258]+[348]+[169]+[357]+[249]+[178]+[456]$$
$$-a_1[257]-a_2[247]+a_3[148]-a_4[147]$$
$$+a_5[235]+a_6[145]+a_8[134]+a_{10}[123].$$

Moreover, two such trivectors are projectively S-equivalent iff the corresponding (C,p) are isomorphic. (Removing "projective" involves fixing a nonzero tangent vector at p.)

(Caveat: This form is not at all unique, so I had to use other means for computing invariants to get things to match up and find it; it's easier to verify once found, though.)

Relation to 3-Selmer groups: Since any stable trivector can be put uniquely into that form, the corresponding scheme of flags is a torsor over $\mathrm{Stab}_{\mathrm{SL}_9/\mu_3}(\gamma)$, an abelian group scheme of order 3^4.* The stabilizer of the trivector corresponding to a curve is $J(C)[3]$, so the scheme of flags is a torsor over such a group. (The order follows from flatness; the identification with $J(C)[3]$ uses the relation to Jacobians.)

*In characteristic 2, this sentence is incorrect in two different ways, but two wrongs make a right.

Given a trivector in that form, we can explicitly compute a suitable affine patch of the rank 4 locus (coming from an induced filtration on V_9). Moreover, it is then not too hard to find an isomorphism between this and $J(C)\backslash\Theta$. (The filtration in the picture means that the otherwise highly nonlinear problem reduces to a very simple nonlinear problem and a lot of linear problems. And we can look for (and find) an isomorphism between the two total spaces that respects the coefficients of the curve.)

This is not *quite* enough to prove things, though: the computation leaves open the possibilities that the compactifications differ. Luckily, both surfaces are known to be abelian*, so we can finesse this.

*In characteristic 0, but finite characteristic follows

Something similar works for the Spin_{16} case, with the $\wedge^4(V_8)$ case following by restricting to suitable nodal curves. Here, we don't know a priori that the compatification is abelian, but we can deal with this by looking at moduli spaces of *vector bundles*: for rank 2 vector bundles with determinant $\propto p$, the residual group is basically SO_4, so we can still explicitly compute invariants. We can then use the incidence relation between two adjacent instances to fill in the boundary.*

Note that a point in the Kummer of a Prym maps to an isomorphism class of rank 2 vector bundles, so the Prym Kummers are Kummers of Jacobians of curves arising from different Vinberg representations (in the centralizer of the appropriate $\mu_2 \subset E_8$).

*Caveat: There may be some details that have not been worked out here.

Of course, we would prefer a less computational approach, and one that actually constructs the trivector rather than simply expresses the trivector in terms of invariants.

This can be done! Consider the Poincaré divisor $X \subset J(C)^2$ (i.e., $\mathrm{Hom}(\mathcal{L}_1, \mathcal{L}_2(p)) \neq 0$). The line bundle on $J(C)^2$ given by $3\Theta_1 + 3\Theta_2 - X$ has 9 global sections (line bundles on abelian varieties are easy!), so X satisfies 9 bilinear equations in $\mathbb{P}^8 \times \mathbb{P}^8$. In fact, these equations are alternating (since p is a Weierstrass point, X contains the diagonal of $J(C)$), so we get a map

$$W_9^* \to \wedge^2 V_9.$$

or an element of $W_9 \otimes \wedge^2 V_9$.

Lemma. There is an isomorphism $W_9 \cong V_9$ such that this element lies in $\wedge^3 V_9$.

Proof: There is a unique Heisenberg-equivariant isomorphism (even in characteristic 3), a Heisenberg-invariant element of $V_9 \otimes \wedge^2 V_9$ generically lies in $\wedge^3 V_9$, and this condition is closed. $\square$

We further find that $J(C)$ lies in the rank 4 locus of this trivector (coming from the 4 sections of 3Θ that vanish on Θ), so this is the trivector we want! (We can show this is unique when C is smooth, and the conditions on the morphism $W_9 \to V_9$ are linear, so easy to solve.)

This is still somewhat unsatisfactory, since we don't get a trivector directly as a trivector. A nicer construction comes from the following two observations:

(a) (Ortega, Minh) The moduli space of rank 3 vector bundles with trivial determinant is (in char. 0) a double cover of $\mathbb{P}^8 = \mathbb{P}(\Gamma(3\Theta)^*)$ ramified along a sextic hypersurface (with dual the Coble cubic).

(b) We can construct such a double cover from a trivector (with the cubic corresponding to the rank 6 locus of the trivector).

Where does (b) come from? A point in $\mathbb{P}(V_9^*)$, i.e., a 1-dimensional subspace of V_9, takes any element of $\wedge^3(V_9)$ to an element of $\wedge^3(V_9/V_1) \cong \wedge^3(V_8)$ What does the generic element of $\wedge^3(V_8)$ look like? There is a natural alternating trilinear form on $\mathfrak{gl}_n$ for any n, given by

$$(A, B, C) \mapsto \mathrm{Tr}(A[B, C]) = \mathrm{Tr}(ABC) - \mathrm{Tr}(ACB),$$

which vanishes on 1, so induces a form on $\mathfrak{pgl}_n$ (i.e., an element of $\wedge^3(\mathfrak{sl}_n)$). For $n > 2$, the stabilizer in $\mathrm{GL}(\mathfrak{sl}_n)$ of this trivector is $\mathrm{Aut}(\mathfrak{sl}_n)$, so for $n = 3$, its $\mathrm{GL}(\mathfrak{sl}_3)$-orbit in $\wedge^3(\mathfrak{sl}_3)$ has dimension $\binom{8}{3}$, so is dense!

It follows with a small amount of additional work that there is a unique invariant of degree 16, and that invariant is a square modulo 4, so adjoining its square root (and normalizing at 2) gives a natural double cover of $\wedge^3(V_8)$,* which can be pulled back to give a double cover of $\mathbb{P}(V_9^*)$ associated to any $\gamma \in \wedge^3(V_9)$.

*This double cover exists since $\mathrm{Aut}(\mathfrak{sl}_3)$ has two components.

This leads to a construction: Given a rank 3 vector bundle V on C with $\det(V) \cong \mathcal{O}_C$, the cotangent space of the moduli space is $\Gamma(\mathfrak{pgl}(V) \otimes \omega_C)$, and the natural form gives

$$\wedge^3(\Gamma(\mathfrak{pgl}(V) \otimes \omega_C)) \to \Gamma(\omega_C^3) \to \Gamma(\omega_C^3|_p) \cong k.$$

One can show that this is anti-invariant under the appropriate involution, so descends to a section of $\wedge^3(\mathcal{T}_{\mathbb{P}^8})(-3)$. (To be precise, it only works over the stable locus, but the complement has codimension ≥ 2.) And there is a natural isomorphism

$$\Gamma(\wedge^3(\mathcal{T}_{\mathbb{P}(V_9^*)})(-3)) \cong \wedge^3(V_9) \ldots$$

Something similar *should* hold for (C,p) with $K_C = 6p$. The analogue of $\wedge^3(V_8)$ is the half-spin representation of Spin_{14}, which again has disconnected stabilizer $(\mathbb{Z}/2\mathbb{Z} \wr G_2)$. So there's a natural double cover of that representation, and thus any element of the half-spin representation of Spin_{16} induces a double cover of the appropriate quadric.

There is a conjecture of Oxbury and Ramanan that associates a modular interpretation of a quartic hypersurface in $\mathbb{P}^{15}$ to a general trigonal curve of genus 4. This has the same Hilbert series as our double cover (of a quadric in $\mathbb{P}^{15}$), and our curves are special, so presumably it's the right interpretation.

So the conjecture for our case is: This double cover is the moduli space of torsors over the group scheme obtained by twisting Spin_8 over the Galois closure of the trigonal structure on C.