Prym varieties in genus four

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Overview

Inspiration: W. P. Milne, *Sextactic Cones and Tritangent Planes of the Same System of a Quadri-Cubic Curve*, Proc. London Math. Soc. (2) 21 (1923), 373–380.

Milne gives a completely synthetic-geometric construction relating tritangents of a genus 4 curve to bitangents of a genus 3 curve...

Interest: Prym varieties form important examples of principally polarized abelian varieties, besides Jacobians.

Of particular importance: Genus 4 is the highest genus where Prym varieties turn out geometrically as Jacobians. (and we know how to construct the corresponding curves for genus $1, 2, 3$ ...)

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Prym varieties: concrete example

Consider a hyperelliptic curve of genus g given by:

$$C\colon y^2 = f_1(x)f_2(x)$$

with $f_1, f_2 \in k[x]$ square-free, even degree, and coprime.

Unramified double cover:

$$\tilde{C}\colon \begin{cases} y_1^2 = f_1(x) \\ y_2^2 = f_2(x) \end{cases}, \quad \pi\colon \begin{array}{ccc} \tilde{C} & \to & C \\ (x, y_1, y_2) & \mapsto & (x, y) = (x, y_1 y_2) \end{array}$$

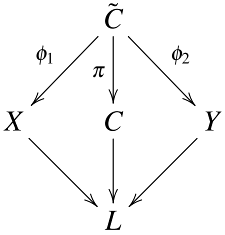
Induced map: $\pi_*\colon \mathrm{Jac}(\tilde{C}) \to \mathrm{Jac}(C)$

Interesting fact: $\ker(\pi_*)$ has two connected components. We define $\mathrm{Prym}(\tilde{C}/C)$ to be the connected component containing 0.

Polarization: $\mathrm{Prym}(\tilde{C}/C)$ has a principal polarization.

Dimensions: $\dim J(\tilde{C}) = 2g-1$ and $\dim \mathrm{Prym}(\tilde{C}/C) = g-1$.

Description of a hyperelliptic Prym



$C\colon y^2 = f_1(x) f_2(x)$

$X\colon y_1^2 = f_1(x)$

$Y\colon y_1^2 = f_2(x)$

$\tilde{C} = X \times_L Y$

Exact sequence induced on Jacobians:

$$\mathrm{Jac}(X) \times \mathrm{Jac}(Y) \xrightarrow{\phi_1^* + \phi_2^*} \mathrm{Jac}(\tilde{C}) \xrightarrow{\pi_*} \mathrm{Jac}(C)$$

For hyperelliptic curves:

- $\mathrm{Prym}\,\tilde{C} \simeq \mathrm{Jac}(X) \times \mathrm{Jac}(Y)$.
- Prym varieties can be described in terms of Jacobians
- $\tilde{C}$ actually covers the associated curves.

Why bother with Prym varieties?

Given an unramified double cover $\pi\colon \tilde{C} \to C$; $\mathrm{genus}(C) = g$

Isogeny Decomposition: $\mathrm{Jac}(\tilde{C}) \simeq \mathrm{Jac}(C) \times \mathrm{Prym}(\tilde{C}/C)$

- We have a map $\tilde{C} \to \mathrm{Prym}(\tilde{C}/C)$.
- $\mathrm{genus}(\tilde{C}) = 2g-1$; $\dim \mathrm{Prym}(\tilde{C}/C) = g-1$
- Interesting arithmetic applications, for instance for determining $\tilde{C}(\mathbb{Q})$.

Pryms as Jacobians:

- For C hyperelliptic, $\mathrm{Prym}(\tilde{C}/C) = \mathrm{Jac}(X) \times \mathrm{Jac}(Y)$.
- $\mathrm{Prym}(\tilde{C}/C)$ is a principally polarized abelian variety.
- Not all PPAVs of dimension > 3 are (twists of) Jacobians.
- Over non-algebraically closed base fields, not all PPAVs of dimension 3 are Jacobians.

Question: If $\mathrm{Prym}(\tilde{C}/C) = \mathrm{Jac}(X)$, how is $\tilde{C}$ related to X?

Arithmetic subtleties

- Let k be a (non-algebraically closed); $\mathrm{char}(k) \neq 2$.
- Let C be a smooth proj. abs. irred. curve over k.
- $\varepsilon \in \mathrm{Pic}(C)[2]$ determines an unramified double cover $\pi\colon \tilde{C} \to C$, but not uniquely so:

$$H^1(k, \mathrm{Aut}(\tilde{C}/C)) = k^\times/k^{\times 2}$$

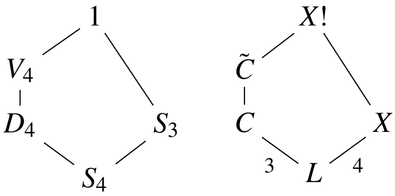
Hyperelliptic example: Factorization $f(x) = f_1(x)f_2(x)$ determines ε. For $\delta \in k^\times$ (modulo $k^{\times 2}$):

$$C\colon y^2 = f_1(x)f_2(x)$$
$$X_\delta\colon y_1^2 = \delta f_1(x)$$
$$Y_\delta\colon y_1^2 = \delta^{-1} f_2(x)$$
$$\tilde{C}_\delta = X_\delta \times_L Y_\delta$$

$$\mathrm{Prym}(\tilde{C}_\delta/C) \simeq \mathrm{Jac}(X_\delta) \times \mathrm{Jac}(Y_\delta)$$

Trigonal construction – Recillas

Galois theory:



Theorem (Recillas): $\mathrm{Jac}(X) = \mathrm{Prym}(\tilde{C}/C)$, so Jacobians of tetragonal curves are Pryms.

In the opposite direction: Let $C \to L$ be trigonal; let $\tilde{C} \to C$ be an unramified double cover.

- Galois closure $\tilde{C}!$ of $\tilde{C} \to L$ generically has group $(C_2)^3 \rtimes S_3 = C_2 \times S_4$;
- Center interchanges geometric components.

Theorem: Given C trigonal and $\tilde{C} \to C$ unramified of degree 2, then there is a *twist* such that $\mathrm{Prym}(\tilde{C}/C) = \mathrm{Jac}(X)$.

Review of genus three

Smooth plane quartic:

$$C\colon\ Q_1(x,y,z)Q_3(x,y,z) = Q_2(x,y,z)^2$$

Double cover:

$$\tilde{C}_\delta\colon\ \begin{cases} Q_1(x,y,z) = \delta u^2 \\ Q_2(x,y,z) = \delta uv \\ Q_3(x,y,z) = \delta v^2 \end{cases}$$

Note: Genus 3 curves are trigonal precisely when they have a rational point, but $\mathrm{Prym}(\tilde{C}_\delta/C)$ must be a Jacobian in all cases.

Special divisor classes: $X \subset W_4^1 \subset \mathrm{Pic}^4(\tilde{C})$

$$X_\delta\colon\ t^2 = -\delta \det(Q_1 + 2sQ_2 + s^2 Q_3)$$

Result:

$$\mathrm{Prym}(\tilde{C}_\delta/C) = \mathrm{Jac}(X_\delta)$$

Genus 4 curves

Reminder: Let C be non-hyperelliptic of genus 4.
Canonical model:

$$\Gamma = Q = 0, \text{ where } \deg(\Gamma) = 3, \ \deg(Q) = 2.$$

General case: Q nonsingular. Two rulings parametrized by a plane conic L; possibly conjugate. This gives two trigonal maps $C \to L$.

Special case: Q is singular. Then Q is a cone over a plane conic L. Gives a trigonal map $C \to L$: vanishing theta null.

Cubics: Span of cubics containing C: $\langle \Gamma, x_0 Q, x_1 Q, x_2 Q, x_3 Q \rangle$.

Generically: Can force 4 singularities on Γ.

Singular cubics

Cayley cubic: Four nodes; admits a *symmetric* presentation:

$$\Gamma_\varepsilon : xyz + xyw + xzw + yzw = \det \begin{pmatrix} x+w & w & w \\ w & y+w & w \\ w & w & z+w \end{pmatrix}$$

Points on Γ_ε parametrize singular conics: pairs of lines.

Double cover: $\tilde{\Gamma} \to \Gamma$ labels these pairs of lines; unbranched outside the nodes.

Theorem: Any cubic Γ over k that has a symmetrization over k^{alg} has one over k (more later).

Theorem (Catanese, B-Sertöz): The double covers of C (modulo twists) correspond exactly to symmetrized cubics containing C:

$$\varepsilon \in \mathrm{Pic}^0(C)[2] \setminus \{0\} \longleftrightarrow \{\text{Symmetrized cubics } \Gamma_\varepsilon \supset C\}$$

Sketch of symmetrization argument

- Suppose $\Gamma \subset \mathbb{P}^3$ is a cubic surfaces with $\mathrm{Sing}(\Gamma) = \mathrm{Spec}(R_\Gamma)$ separated of degree 4, not contained in a plane.
- Check that if $R_{\Gamma_1} \simeq R_{\Gamma_2}$ (as k-algebras) then $\Gamma_1 \simeq \Gamma_2$ (over k)
- Choose $R = k[\theta]$ with $f(\theta) = a_0 + a_1\theta + \cdots + a_3\theta^3 + \theta^4 = 0$
- Consider cubic threefold:

$$\mathscr{D}' : \det \begin{pmatrix} u_0 & u_1 & u_2 \\ u_1 & u_2 & u_3 \\ u_2 & u_3 & u_4 \end{pmatrix} = 0$$

- Construct $\Gamma_f = \mathscr{D}' \cap \{a_0u_0 + \cdots + a_3u_3 + u_4 = 0\}$.
- Check that $\mathrm{Sing}(\Gamma_f) \simeq \mathrm{Spec}(R)$.

Note: We can take f non-separable and obtain special Γ.

Double covers of genus 4 curves

(assume Γ_ε sufficiently general for now)
Parametrization: Symmetrization induces a birational map

$$\mathbb{P}^2 \to \Gamma_\varepsilon$$

(take the conic with a singularity at the given point)

Distinguished double cover:

$$\begin{array}{ccc} \tilde{C}_\varepsilon & \longrightarrow & \tilde{\Gamma}_\varepsilon \\ \downarrow & & \downarrow \\ C & \longrightarrow & \Gamma_\varepsilon \end{array}$$

Question: Do we have $\mathrm{Prym}(\tilde{C}_\varepsilon/C) = \mathrm{Jac}(X_\varepsilon)$ and if so, how do we construct X_ε?

Dual varieties

Recall: Given a surface $V : f(x,y,z,w) = 0$, we have a rational map:

$$\mathbb{P}^3 \to \widehat{\mathbb{P}}^3; \quad (x:y:z:w) \mapsto \left(\frac{\partial f}{\partial x} : \frac{\partial f}{\partial y} : \frac{\partial f}{\partial z} : \frac{\partial f}{\partial w}\right)$$

Dual variety: Image $\widehat{V}$ under this map.

- If Γ_ε is a Cayley cubic then $\widehat{\Gamma}_\varepsilon$ is a quartic surface.
- For a nonsingular quadric Q, we have that $\widehat{Q}$ is an isomorphic quadric.

Define: $X_\varepsilon = \widehat{\Gamma}_\varepsilon \cap \widehat{Q}$

- Rulings on $Q \simeq \widehat{Q}$ give tetragonal maps $X_\varepsilon \to L$.
- Symmetrization gives rise to a map $\mathfrak{q}_\varepsilon : \mathbb{P}^2 \to \widehat{\Gamma}_\varepsilon$.
- Compatible with $\mathbb{P}^2 \to \Gamma_\varepsilon \to \widehat{\Gamma}_\varepsilon$
- $\mathfrak{q}_\varepsilon^{-1}(X)$ a smooth plane quartic (a canonical model of X).

Corollary: $\mathrm{Jac}(X_\varepsilon) = \mathrm{Prym}(\tilde{C}_\varepsilon / C)$.

Reconstruction from X

Prym data: (C, ε) with $\varepsilon \in \mathrm{Pic}(C)[2]$.

Arithmetic subtlety: Generally, even if $\varepsilon \in \mathrm{Pic}(C)$ and $\varepsilon \otimes \omega_C$ is effective and Galois stable, it may not admit an effective representative over k (think $\mathrm{Pic}^1(L)$ for a conic L):

Theorem: For $\varepsilon \in \mathrm{Pic}(C)[2]$ as above, $|\varepsilon \otimes \omega_C| \simeq \mathbb{P}^2$.
(thanks to symmetrization of Γ_ε)

Data obtained: $(X, \mathscr{L}_1, \mathscr{L}_2)$, with tetragonal pencils $\mathscr{L}_1, \mathscr{L}_2 \in \mathrm{Pic}(C)$ and $\mathscr{L}_1 \otimes \mathscr{L}_2 \simeq \omega_X^{\otimes 2}$.

Equivalently: Pair $\{\mathscr{L}_1 \otimes \omega_X^\vee, \mathscr{L}_2 \otimes \omega_X^\vee\}$ amounts to specifying a point $\kappa_\varepsilon \in \mathrm{Kum}(X)$.

Correspondence: (See [Recillas, Hidalgo-Recillas])

$$(C, \varepsilon) \leftrightarrow (X, \kappa_\varepsilon)$$

Degeneracies of Q

If Q is singular, then C has a *vanishing theta null*:
$\theta_0 \in \mathrm{Pic}(C)$, with $2\theta_0 \simeq \omega_C$ and $\ell(\theta_0) = 2$.

Odd case: $\ell(\theta_0 \otimes \varepsilon) = 1$
Tritangent plane H_ε. Then $\Gamma_\varepsilon = H_\varepsilon \cup Q$. Construction of X breaks down.

Link with del Pezzo surfaces:

- C is part of fixed locus of Bertini involution on a dP1 with a marked exceptional curve E.
- Blowing down E gives a dP2 with a point P
- dP2 has a Geiser involution; fixes a genus 3 curve X
- Projection from P gives a tetragonal map $X \to \mathbb{P}^1$.

Correspondence: This leads to (X, κ_ε) with $\kappa_\varepsilon = 0$.

Degeneracies of Q (continued)

Even case: $\ell(\theta_0 \otimes \varepsilon) = 0$ (Q still a cone).

- Plane $H \subset \widehat{\mathbb{P}}^3$ dual to vertex
- Conic $q \subset H$ for tangent planes
- $\widehat{Q} = q \subset H$.
- Γ_ε still normal.
- $\widehat{\Gamma}_\varepsilon \cap \widehat{Q}$ marks 8 points on a conic
- X is hyperelliptic.

Note: In this case X admits quadratic twists, so $\mathrm{Prym}(\tilde{C}'/C)$ is a Jacobian for twists of $\tilde{C}_\varepsilon/C$ as well.

Degeneracy of Γ_ε

Bielliptic case:

- Suppose $\mathfrak{b}\colon C \to E$ is a double cover of a plane cubic curve
- Suppose $\varepsilon' \in \mathrm{Pic}(E)[2]$ and $\varepsilon = \mathfrak{b}^*(\varepsilon')$.
- Then Γ_ε is a cone over E (admitting 3 distinct symmetrizations)
- Let H be the plane dual to the vertex of Γ_ε.
- Then $L = H \cap \widehat{Q}$ is a conic
- Symmetrization of Γ_ε leads to a $4:1$ map $\mathbb{P}^2 \to H$, and hence a tetragonal map $X \to L$.

Correspondence: We get that $\mathscr{L}_1 \simeq \mathscr{L}_2$, so κ_ε is in the image of $\mathrm{Jac}(X)[2]$.

Arithmetic: $C \to E$ admits quadratic twists. These are not reflected in $\mathrm{Prym}(\tilde{C}_\varepsilon/C)$.

Moduli space stratification

Prym curves: $\mathscr{R}_4$: Curves C with $\varepsilon \in \mathrm{Pic}(C)[2]$.

Strata:

$\mathscr{R}_4^{\mathrm{o}}$	non-hyperelliptic curves, excluding below
$\mathscr{R}_4^{\mathrm{odd}}$	curves with vanishing theta-null and $\varepsilon \otimes \theta_0$ odd,
$\mathscr{R}_4^{\mathrm{even}}$	curves with vanishing theta-null and $\varepsilon \otimes \theta_0$ even,
$\mathscr{R}_4^{\mathrm{biell}}$	bielliptic (C, ε) without vanishing theta-null,
$\mathscr{R}_4^{\mathrm{biell,even}}$	bielliptic (C, ε) with vanishing theta-null.

(bielliptic and odd does not occur)

Correspondence:

stratum	Q_C	Γ_ε	X_ε	κ_ε
$\mathscr{R}_4^{\mathrm{o}}$	nonsingular	irreducible	non-hyp.	general
$\mathscr{R}_4^{\mathrm{even}}$	cone	irreducible	hyperelliptic	general
$\mathscr{R}_4^{\mathrm{biell}}$	nonsingular	cone	non-hyp.	self-residual
$\mathscr{R}_4^{\mathrm{biell,even}}$	cone	cone	hyperelliptic	self-residual
$\mathscr{R}_4^{\mathrm{odd}}$	cone	reducible	non-hyp.	canonical