A scalar conservation law with discontinuous flux for supply chains with finite buffers.

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Collaborators

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Based on a MA thesis of P. Goossens, TU Eindhoven.

[2] D. Armbruster, S. Goettlich, M. Herty - A continuous model for supply chains with finite buffers, SIAM J. on Applied Mathematics (SIAP), Vol. **71**(4), pp. 1070-1087, (2011).

Continuous models of production lines I[3]

Usual model: Faithful representation of the factory using *Discrete Event Simulations*, e.g. χ (TU Eindhoven)

Problem:

Simulation of production flows with stochastic demand and stochastic production processes requires Monte Carlo Simulations

Takes too long for a decision tool

[3] Dieter Armbruster, Daniel Marthaler, Christian Ringhofer, Karl Kempf, Tae- Chang Jo: Operations Research **54**(5), 933-950, 2006

Fundamental Idea:
Model high volume, many stages, production via a fluid.

Basic variable

product density (mass density) $\rho(\mathbf{x}, \mathbf{t})$.
x- is the position in the production process, $x \in [0, 1]$.
- degree of completion
- stage of production

Mass conservation and state equation

$$\begin{aligned} \frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} &= 0 \\ F &= \rho v_{eq} \end{aligned}$$

Typical models for the equilibrium velocity v_{eq} (state equation) are

$$\begin{aligned} v_{eq}^{traffic}(\rho) &= v_0\left(1 - \frac{\rho}{\rho_c}\right) \\ v_{eq}^{Q} &= \frac{\mu}{1+L} \\ v_{eq} &= \Phi(L) \end{aligned}$$

with L the total load (WIP) given as $L(\rho) = \int_0^1 \rho(x,t)dx$

Note: $\Phi(L)$ may be determined experimentally or theoretically

Production lines with finite buffers

Current assumption

Buffers can become infinite
ρ can have δ-measures
Flux may be restricted but not density

Production lines for larger items, e.g. cars

There exists only a small buffer between machines
Need to implement a limit on ρ in our model

Simulation[4]

Experiment

100 identical machines with capacity $\mu = 1$
all buffers between machines have identical capacity of M

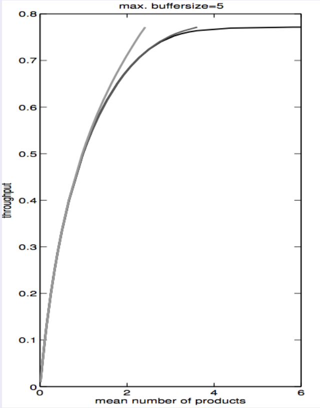
1. fill an empty factory with a constant influx rate $\lambda < 1$
2. shut down the last machine
3. factory fills up and stops working when the first buffer is at its maximum.
4. restart last machine and drain the factory until it reaches steady state again.

Show movie_high and movie_low

[4] P. Goossens, Modeling of manufacturing systems with finite buffer sizes using PDEs, Masters Thesis, TU Eindhoven, 2007

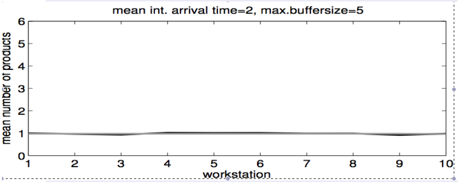
Model needs to explain:

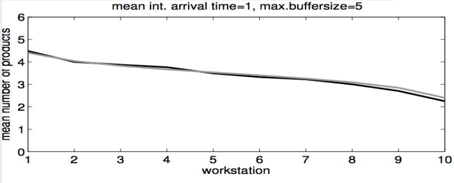
- The maximal steady state throughput λ_{max} of the production line is much lower than 1



More:

- The steady–state WIP distribution $\rho_{ss}(x)$ for $\lambda << 1$ is constant in x
- The steady–state WIP distribution $\rho_{ss}(x)$ for $\lambda \approx \lambda_{max}$ decays almost linearly in x





More:

- At shut down, the production line is filled up by a backwards moving wave.
  wave speed is

$$v_{shutdown} = \frac{\lambda}{M - \int_0^1 \rho_{ss}(x)dx}. \quad (1)$$

- The transient drain depends on the influx λ.
  - If $\lambda \approx \lambda_{max}$ then the factory drains from the end.
  - If $\lambda < \lambda_{max}$ then WIP is reduced by a wave "eating" into it from upstream and at the same time WIP uniformly drains downstream.

Phenomenology IV

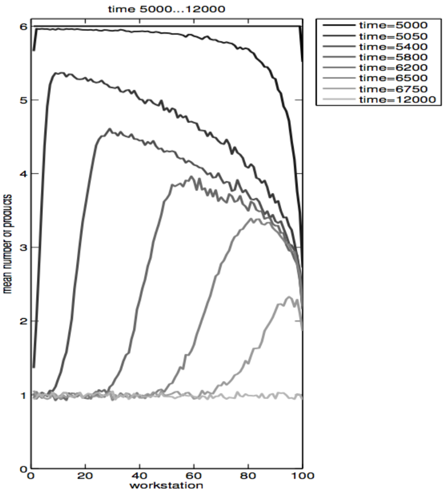


Figure: $\lambda < \lambda_{max}$. The WIP distribution drifts downwards and "gets eaten" from the back

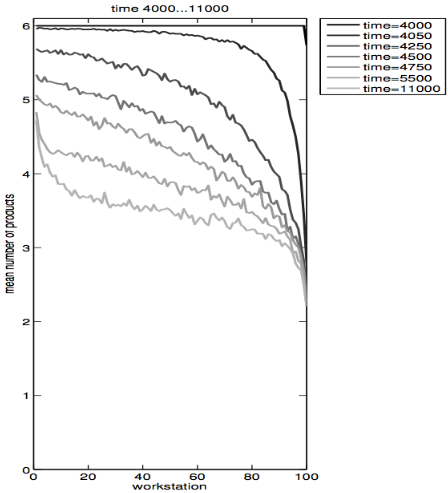


Figure: $\lambda \approx \lambda_{max}$, the system approaches the steady state distribution almost uniformly in space.

Two fundamental stochastic processes

- The production process with mean processing rate $\mu = 1$.
- The blocking process when the buffer becomes full.

Probability for machine idling

- due to starvation - i.e. nothing is in the queue.
  Basic assumption: $M/M/1$ queue
- due to blocking. Probability for this to occur will increase with the distance from the end of the supply chain.

Inhomogeneous processing rate II

Together they lead to an inhomogeneous processing rate

$$\tilde{\mu} = c(x)\mu.$$

We make three assumptions for $c(x)$;

- $c(1) = 1$.
- $c(x)$ linearly increases with the steady state influx λ.
- $c(x)$ linearly increases as a function of x.

Consistent Assumption:

$$\tilde{\mu} = c(x)\mu = \lambda m(\rho)(x - 1) + \mu$$

Inhomogeneous and discontinuous flux

Assumptions:

- machine process is Poisson, i.e.

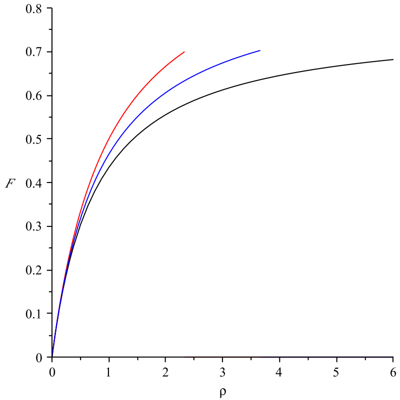
$$\lambda = \frac{\mu}{1+\rho}$$

- $m(\rho)$ is linear in ρ, i.e.

$$m(\rho) = k\rho$$

Flux function

$$F(\rho, x) := \begin{cases} \frac{\mu\rho}{1+\rho+\rho(1-x)} & \text{for } \rho < M \\ 0 & \text{for } \rho \geq M. \end{cases} \quad (2)$$



Steady state WIP distribution

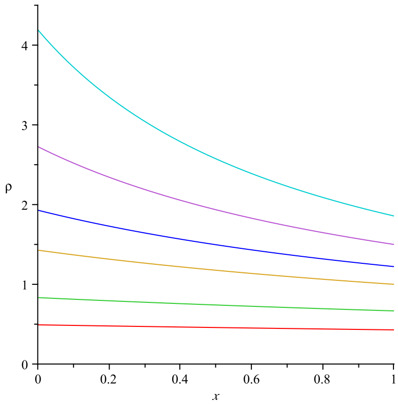
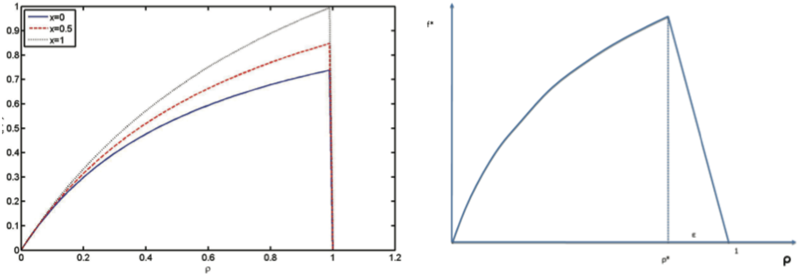


Figure: Steady states for a flux function (2) and different values for the inflow densities λ

Riemann problem

For different initial conditions we get different kinetic waves:

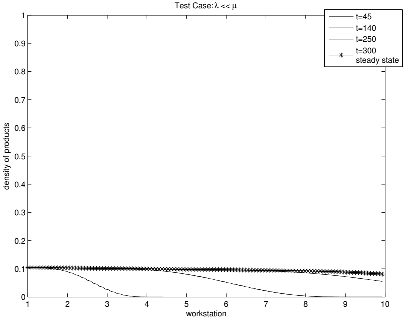
- a rarefaction - speed $\lambda = f'(\rho)$. Filling wave - start at a traffic light.
- a shock wave - speed $s = \frac{f(\rho_l)}{\rho_l - M}$. Blocking wave.
- a shock wave traveling with infinite speed. Information wave after restart.
- This wave is followed by a classical rarefaction wave emanating at $x = 1$ and a shock wave emanating at $x = 0$



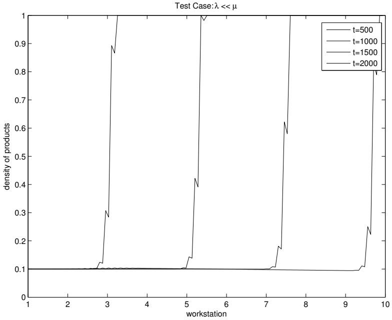
(a) flux model

(b) smoothing

Figure: Implementations

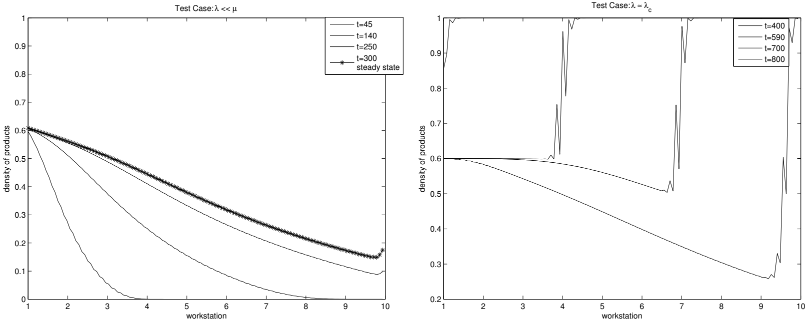


(a) filling an empty factory



(b) blocking

Figure: $\lambda < \lambda_{max}$.



(a) filling an empty factory

(b) blocking

Figure: $\lambda \approx \lambda_{max}$., Ramp up and steady state solution

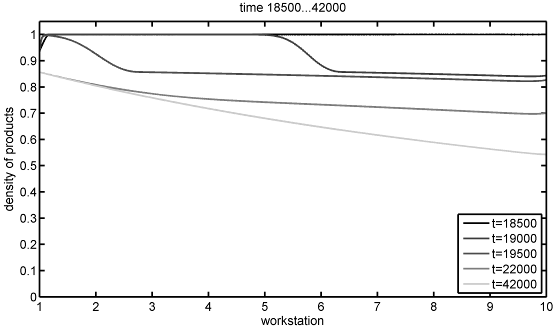


Figure: restarting production again and final equilibrium for $\lambda \approx \lambda_{max}$

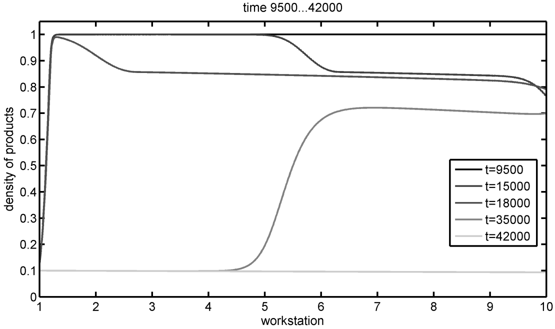


Figure: restarting production again and final equilibrium for $\lambda < \lambda_{max}$.

TASEP[5]

Totally Asymmetric Simple Exclusion Process

- Choose N sites
- Each site can only be occupied by one particle
- A particle can move to the right only if the site next to it is free
- Update is time discrete and via random choice of pairs of sites $(i, i+1)$

[5] G. Schütz and E. Domany, Journal of Statistical Physics, Vol. 72, Nos. 1/2, 1993

Update rules

Define $\tau_i(t) = 1$, if site i is occupied at time t, otherwise $\tau_i(t) = 0$
Randomly choose a pair of indices $i, i+1$ and follow the update rules:

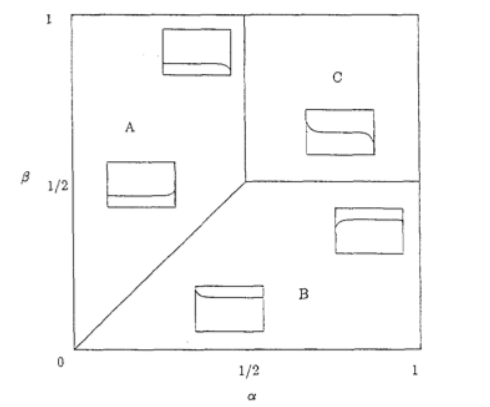
$$
\begin{aligned}
\tau_i(t+1) &= \tau_i(t)\tau_{i+1}(t) \\
\tau_{i+1}(t+1) &= \tau_{i+1} + (1-\tau_{i+1}(t))\tau_i(t) \\
\tau_1(t+1) &= 1 \text{ with probability } \tau_1(t) + \alpha(1-\tau_1(t)) \\
\tau_1(t+1) &= 0 \text{ with probability } (1-\alpha)(1-\tau_1(t)) \\
\tau_N(t+1) &= 1 \text{ with probability } (1-\beta)\tau_N(t) \\
\tau_N(t+1) &= 0 \text{ with probability } 1-(1-\beta)\tau_N(t)
\end{aligned}
$$

In steady state

the flux function for the hydrodynamic limit of TASEP is

$$F = \rho(1-\rho)$$

Phase transition diagram:



Heuristic results

- can model the breakdown of a production line quantitatively
- can model the steady states for a production line with finite buffer with one fitting parameter quantitatively
- can model the resumption of production qualitatively

Single line

- First principle theory - what is different here from TASEP?
- Production planning problem:[6] Find the influx $\lambda(t), t \in [0,\tau]$:s.t.

$$j(\rho,\lambda) = \frac{1}{2}\int_0^{\tau} (F(1,t) - d(t))^2 \; dt$$

is minimal, subject to

$$\frac{\partial \rho}{\partial t} + \frac{\partial F}{\partial x} = 0$$

[6] Michael La Marca, Dieter Armbruster, Michael Herty and Christian Ringhofer: Control of continuum models of production systems, IEEE Trans. Automatic Control **55** (11), p 2511 - 2526 (2010).

Open Problems II

Cascading failures

- Lai et al. determine susceptibility of network structures to cascading failures.
- Redistribution of load leads to a steady state that is a fractured network.
- Correct description for extremely fast transitions through network, e.g. internet.
- Here: Network has a significant travel time.
- Travel time depends on load and on stochasticity in the link

Open problems III

Transients of cascading failures

- New issue: transient time for the network to collapse, τ_C.
- New issue: transient time for the network to repair, τ_R.
- On a linear production line: $\tau_C << \tau_R$.
- Question: Is this true for other network topologies?
- Relevant e.g. for a major shutdown of an airport due to eg. terrorism or weather.
- Opens up the opportunity for countermeasures: Shedding load within τ_C may prevent collapse.