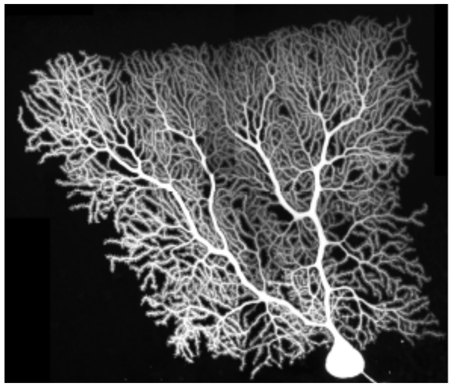
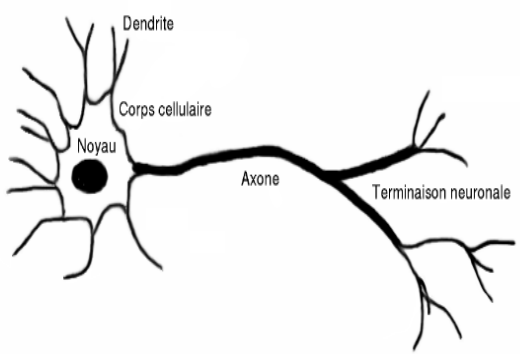
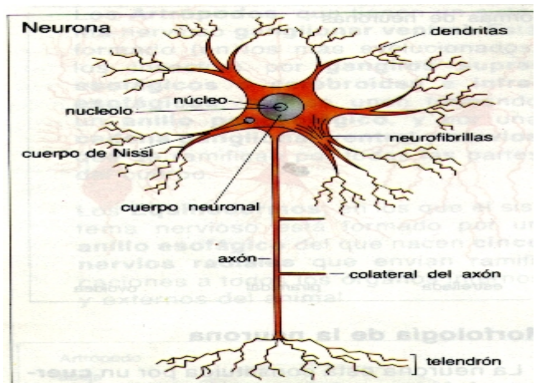
PDE models of neural networks

Benoît Perthame





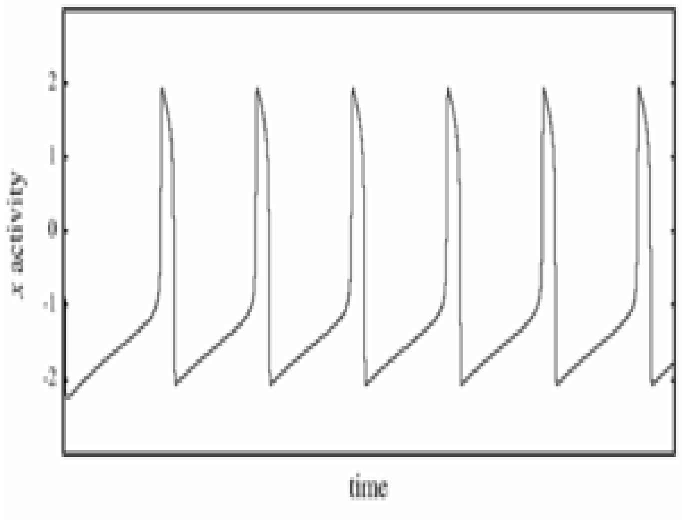


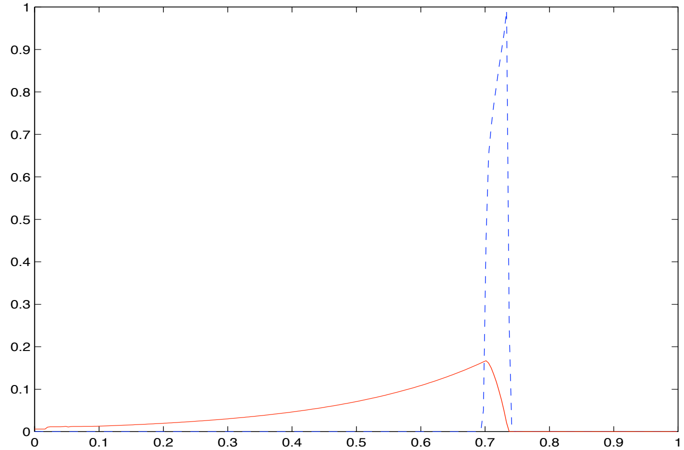
Introduction

The electrically active cells are characterized by an **action potential**

- Hodgkin-Huxley
- FitzHugh-Nagumo
- Morris-Lekar
- Izhikevich
- Mitchell-Schaeffer

Introduction





Solutions to the Hodgkin-Huxley model and to the FitzHugh-Nagumo model

These models are accurate
but very expensive/difficult to use for large assemblies of neurones.

Introduction

The Wilson-Cowan model (1972) describes the firing rates $N(t, x)$ of neuron assemblies located at position x through an integral equation

$$\frac{d}{dt}N(x,t) = -N(x,t) + \int w(x,y)\sigma\Big(N(y,t)\Big)dy + s(x,t)$$

Feature : multiple steady states and bifurcation theory (Bressloff-Golubitsky, Chossat-Faugeras)

- $\sigma(\cdot) =$ sigmoid
- $w_{ij} =$ connectivity matrix
- $s =$ source

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Aim : large scale brain activity, visual hallucinations (Klüver, Oster, Siegel...)

OUTLINE OF THE LECTURE

I. Principle of Noisy Integrate and Fire model

II. The nonlinear Noisy Integrate and Fire model

III. The elapsed time approach

Leaky Integrate and Fire

The Leaky Integrate & Fire model is simpler

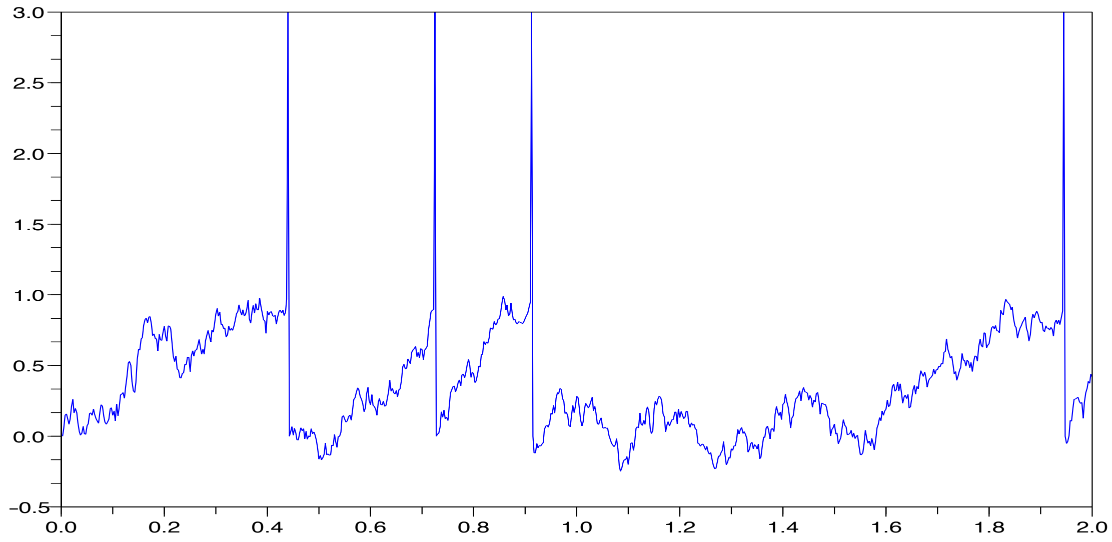
$$dV(t) = \Big(-V(t) + I(t) \Big) dt + \sigma dW(t), \qquad V(t) < V_{\text{Firing}}$$

$$V(t_-) = V_{\text{Firing}} \quad \Longrightarrow \quad V(t_+) = V_{\text{reset}}.$$

The idea was introduced by L. Lapicque (1907).

- $I(t)$ input current
- Noise or not
- Stochastic firing

Leaky Integrate and Fire



Solution to the LIF model

- N. Brunel, V. Hakim, W. Gerstner and W. Kistler...
- Fit to measurements
- Explains qualitatively observations on the brain activity

Leaky Integrate and Fire

Written in terms of PDEs, the probability $n(v,t)$ to find a neuron at the potential v

$$
\begin{cases}
\frac{\partial n(v,t)}{\partial t} + \frac{\partial}{\partial v} \overbrace{\Big[\Big(- v + I(t)\Big) n(v,t)\Big]}^{\text{leak+external currents}} - a \overbrace{\frac{\partial^2 n(v,t)}{\partial v^2}}^{\text{Noise}} = \overbrace{\delta(v = V_R) N(t)}^{\text{neurons reset}}, \quad v \leq V_F, \\
n(V_F, t) = 0, \qquad n(-\infty, t) = 0, \\
N(t) := -a \frac{\partial n(V_F,t)}{\partial v} \geq 0, \quad \text{(the total flux of neurons firing at } V_F).
\end{cases}
$$

$N(t)$ is also a Lagrange multiplier for the constraint

$$\int_{-\infty}^{V_F} n(v,t)dv = 1.$$

Leaky Integrate and Fire

$$
\begin{cases}
\frac{\partial n(v,t)}{\partial t} + \frac{\partial}{\partial v}\left[\Big(-v + I(t)\Big) n(v,t)\right] - a\frac{\partial^2 n(v,t)}{\partial v^2} = \delta(v = V_R) N(t), \qquad v \leq V_F, \\
n(V_F, t) = 0, \qquad p(-\infty, t) = 0, \\
N(t) := -a\frac{\partial n(V_F,t)}{\partial v} \geq 0, \quad \text{(the total flux of firing neurons at } V_F).
\end{cases}
$$

Properties (M. Cáceres, J. Carrillo, BP) The solutions satisfy

- $n \geq 0, \qquad \int_{-\infty}^{V_F} n(v,t)dv = 1,$
- For $I(t) \equiv 0$, $n(v,t) \underset{t\to\infty}{\longrightarrow} P(v)$ the unique steady state of integral 1 (desynchronization)
- The convergence rate is exponential

Leaky Integrate and Fire

The proof uses

- the Relative Entropy

$$\frac{d}{dt}\int_{-\infty}^{V_F} P(v)H\Big(\frac{n(v,t)}{P(v)}\Big)dv \leq 0,$$

for $H(\cdot)$ convex,

- Hardy/Poincaré inequality,

$$\int_{-\infty}^{V_F} P(v)|u(v)|^2 dv \leq C \int_{-\infty}^{V_F} P(v)|\nabla u(v)|^2 dv,$$

for $\int_{-\infty}^{V_F} P(v)u(v)dv = 0$ [notice $P(V_F) = 0$].

Ledoux, Barthe and Roberto

Noisy LIF networks

For networks, the current $I(t)$ is related to the total activity of the network

$$\begin{cases} \frac{\partial n(v,t)}{\partial t} + \frac{\partial}{\partial v}\left[\Big(-v + bN(t)\Big)n(v,t)\right] - a\Big(N(t)\Big)\frac{\partial^2 n(v,t)}{\partial v^2} = \delta_{V_R}(v)N(t), \quad v \leq V_F, \\ n(V_F,t) = 0, \qquad n(-\infty,t) = 0, \\ N(t) := -a\Big(N(t)\Big)\frac{\partial}{\partial v}n(V_F,t) \geq 0, \quad \text{total flux of firing neurons at } V_F. \end{cases}$$

Constitutive laws

- $I(t) = bN(t)$
- $b =$ connectivity
- $b > 0$ for excitatory neurones
- $b < 0$ for inhibitory neurones
- $a(N) = a_0 + a_1 N$

Noisy LIF networks

Theorem (J. Carrillo, BP, D. Smets) Assume

- $a = a_0 > 0$ and $b < 0$ (inhibitory)
- the initial data is bounded by a supersolution (in a certain sense)

Then,

- There are global solutions
- Uniformly bounded for all $t > 0$

Noisy LIF networks

Theorem (M. Cáceres, J. Carrillo, BP)

Assume

- $a \geq a_0 > 0$ and $b > 0$
- the initial data is concentrated enough around $v = V_F$.

Then,

- there are **NO** global weak solutions
- larger nonlinear diffusion does not help

Noisy LIF networks

Theorem (M. Cáceres, J. Carrillo, BP)

Assume

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Then,

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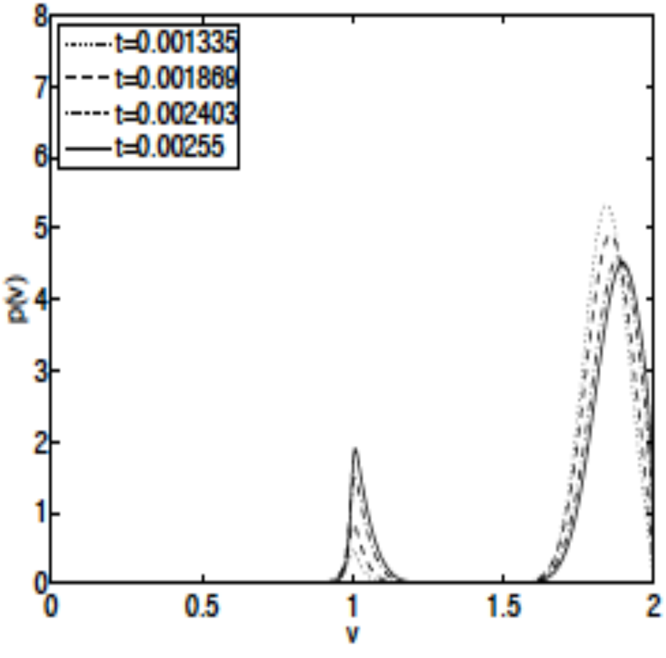
- larger nonlinear diffusion does not help

Possible interpretation

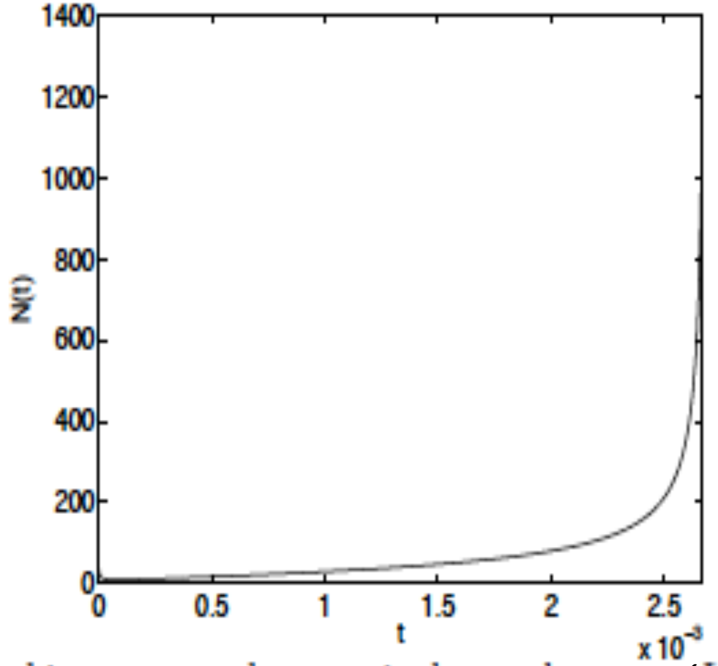
- $N(t) \to \rho\delta(t - t_{\mathrm{BU}})$.
- partial synchronization (see S. Ha)

Noisy LIF networks

Numerical solution of the blow-up phenomena



probability density $n(v)$

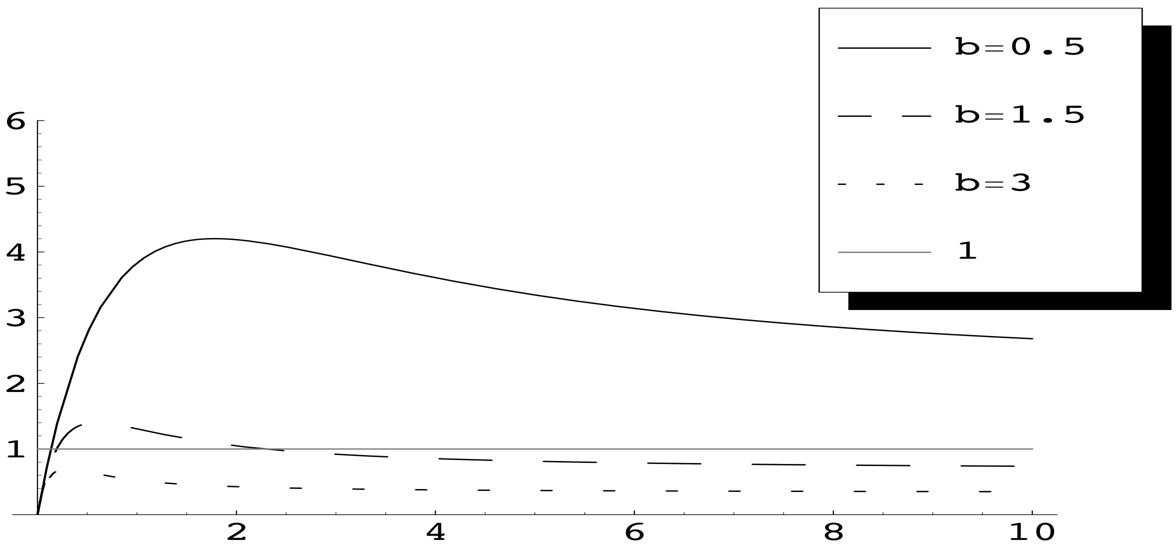


Total neuronal activity $N(t)$

Noisy LIF networks

Theorem (Steady states) For

- $b > 0$ small enough, there is a unique steady state
- $b > (V_F - V_R)$, $2ab < (V_F - V_R)^2 V_R$, then there are at least 2 steady states
- $b > 0$ large enough, there are no steady states.



Noisy LIF networks

Similarity with a Keller-Segel type model

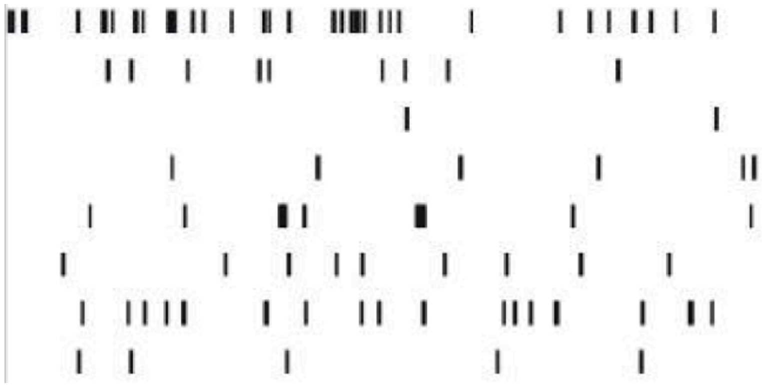
by V. Calvez and R. Voituriez

for microtubules arrangments on the membrane

$$\begin{cases} \frac{\partial n(z,t)}{\partial t} - \frac{\partial}{\partial z}\left[\mu(t)n(z,t)\right] - \frac{\partial^2 n(z,t)}{\partial z^2} = 0, \qquad z \geq 0, \\ \frac{\partial}{\partial z} n(0,t) + \mu(t)n(0,t) = 0, \\ \frac{d\mu(t)}{dt} = n(0,t) - \frac{\mu(t)}{L}. \end{cases}$$

- Blow-up for large mass
- Smooth solutions for small mass (and stable steady state)

Elapsed time structured model



K. Pakdaman, J. Champagnat, J.-F. Vibert have proposed to structure by time rather than potential which is a possible coding of neuronal information

Elapsed time structured model

- s represents the time elapsed since the last discharge
- $n(s,t)$ probability of finding a neuron in 'state' s at time t
- $p(s,N) \leq 1$ represents the firing rate of neurons in the 'state s'
- $N(t)$ = activity of the network + external signaling

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} \quad + \overbrace{\frac{\partial n(s,t)}{\partial s}}^{\text{elapsed time advances}} \quad + \overbrace{p(s, bN(t))\ n(s,t)}^{\text{firing neurons}} = 0, \\ n(s=0,t) = \underbrace{\int_0^{+\infty} p(s, bN(t))\ n(s,t) ds,}_{\text{neurons reset}} \end{cases}$$

This model always satisfies $$\int_0^{+\infty} n(s,t) ds = 1.$$

Elapsed time structured model

- s represents the time elapsed since the last discharge
- $n(s,t)$ probability of finding a neuron in 'state' s at time t
- $p(s,N) \leq 1$ represents the firing rate of neurons in the 'state s'
- being given a total activity N

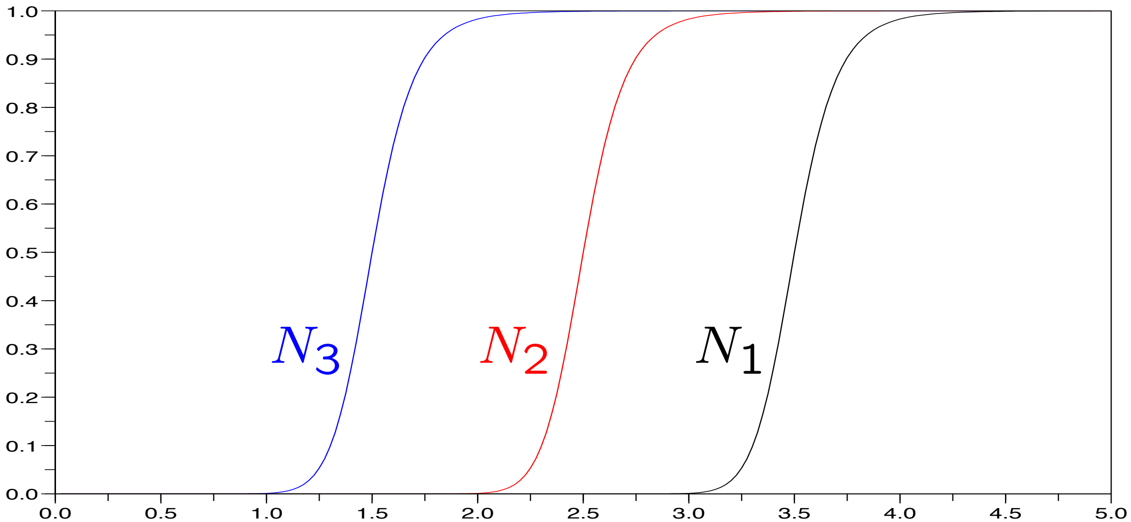
$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + p(s, bN(t))\ n(s,t) = 0, \\ n(s=0,t) = \int_0^{+\infty} p(s, bN(t))\ n(s,t) ds, \end{cases}$$

$$N(t) := n(s=0,t)$$

- $b > 0$ connectivity of the network
- excitatory neurons are represented by $\frac{\partial p(s,N)}{\partial N} > 0$

Elapsed time structured model

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + p(s, bN(t))\ n(s,t) = 0, \\ n(s=0,t) = \int_0^{+\infty} p(s, bN(t))\ n(s,t)ds, \end{cases}$$



the function $s \mapsto p(s,N)$ (refractory state+ fast transition)

Elapsed time structured model

With *synaptic integration*

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + p(s, X(t))\ n(s,t) = 0, \\ N(t) := n(s=0,t) = \int_0^{+\infty} p(s, X(t))\ n(s,t) ds, \end{cases}$$

$$X(t) := b \int_0^t N(t-u)\omega(u) du.$$

Elapsed time structured model

$$\begin{cases} \frac{\partial n(s,t)}{\partial t} + \frac{\partial n(s,t)}{\partial s} + p(s, bN(t))\ n(s,t) = 0, \\ \frac{1}{b} N(t) := n(s=0,t) = \int_0^{+\infty} p(s, bN(t))\ n(s,t) ds, \end{cases}$$

Properties

- $n \geq 0, \qquad \int_0^\infty n(s,t) ds = 1,$
- $N(t) \leq 1$, $n(s,t) \leq 1$,
- there is a unique solution,

Linear case For $p \equiv p(s)$ then

- $n(s,t) \underset{t \to \infty}{\longrightarrow} P(s)$ the unique steady state.

Elapsed time structured model

The proof goes through Generalized Relative Entropy

$$\frac{d}{dt}\int_0^\infty \Phi(s)P(s)H\Big(\frac{n(s,t)}{P(s)}\Big)ds \leq 0,$$

for $H(\cdot)$ convex.

Elapsed time structured model

Properties

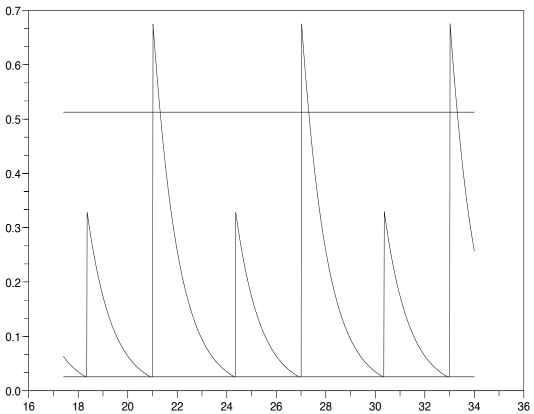
• For small or large connectivity (b small or large) then desynchronization still holds

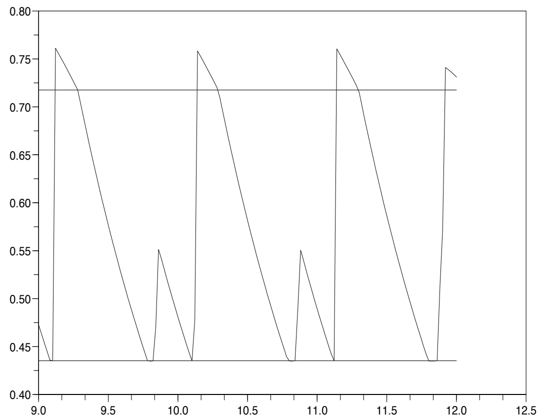
$$n(s,t) \underset{t \to \infty}{\longrightarrow} P_b(s)$$

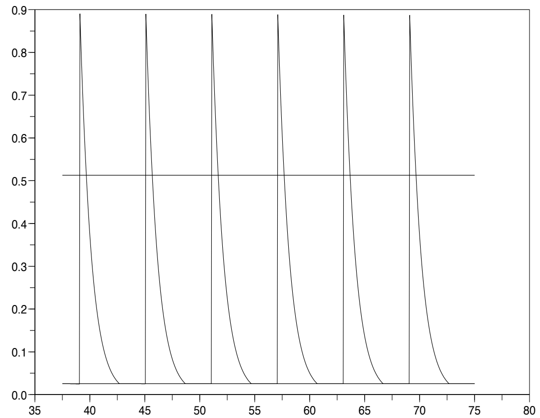
• There are several periodic solutions (explicit),

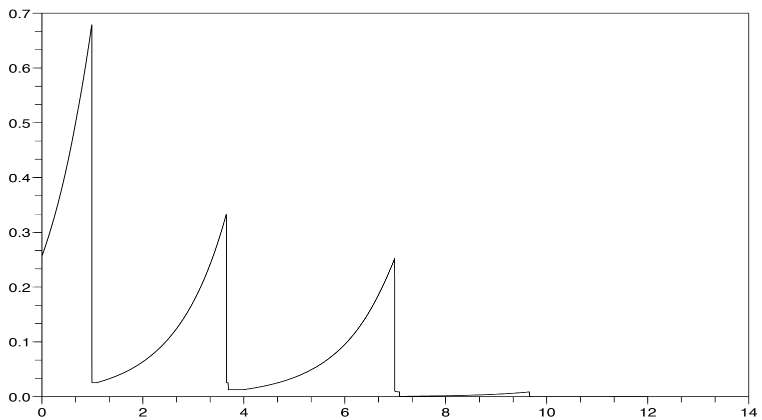
• These are stable (observed numerically).

Elapsed time structured model









Conclusion

THANKS TO MY COLLABORATORS

M. J. Carceres (U.Granada), J. A. Carrillo (U. A. Barcelona)
(for I & F ; J. Mathematical Neurosciences 2011)

D. Smets (work in preparation)

K. Pakdaman, D. Salort (Inst. J. Monod, U. Paris Diderot)
(for elapsed time ; Nonlinearity 2010)

THANK YOU ALL