Remark on a Kinetic Equation for Compton Scattering

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Plan

1. Compton Scattering
2. Boltzmann-Compton equation: known results
3. Kompaneets approximation: known results
4. Zel'dovich approximation
5. Final comments

Compton Scattering

In 1922 (!) A. H. Compton discovers the increase of wavelength of X-rays due to scattering of the incident radiation by free electrons, which implies that the scattered quanta have less energy than the quanta of the original beam. This effect, the Compton effect, illustrates the particle concept of electromagnetic radiation.

In a plasma embedded in a radiation field of temperature T. The scattering of photons by the electrons in the plasma will continuously transfer the energy between the two components.

There are regions in the Universe - like the cluster of galaxies - that contain hot, ionized gas. Cosmic microwave background radiation photons, that fills the Universe, when passes through these regions, they will be scattered by the electrons (which are at a much high temperature) and gain energy. This will distort the cosmic microwave background radiation spectrum in the vicinity of a cluster of galaxies: Sunyaev-Zeldovich effect.

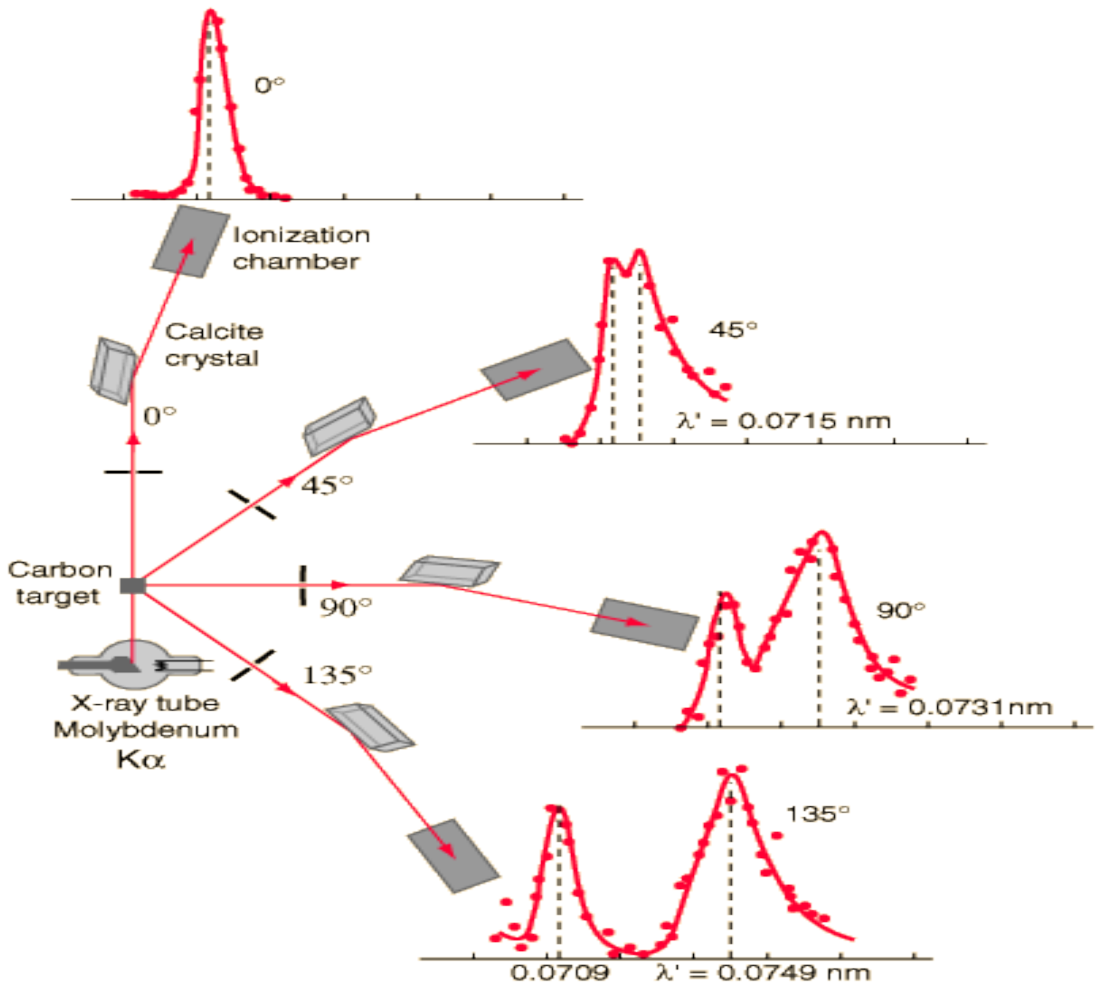

Figure 1: The Compton experiment.

The model

- Photons are quantum particles with no mass.

- The electrons (with mass normalised to $m = 1$) are at non relativistic and classical equilibrium. The density of electrons is given by: $e^{-|p|^2}$ where, if p is the momentum of the electrons, $|p|^2$ is their energy.

- The gas of photons is spatially homogeneous. The density function is then $F(t, p)$: represents the density of photons that at time t have momentum p. The energy of a photon of momentum p is $|p|$.

- Neglect the photon-photon interaction and Bremstrahlung effects. The photon density function $F(t, p)$ satisfies the following equation:

The Boltzmann equation

$$\frac{\partial F}{\partial t}(p,t) = \int_{\mathbb{R}^3} S(p,p')\left(e^{-|p|}F(p',t)(1+F(p,t)) - e^{-|p'|}F(p,t)(1+F(p',t))\right)dp'$$

(A. S. Kompaneets, JETP 1956; H. Dreicer, Phys. Fluids 1964)
If we consider only radially symmetric densities (isotropic gases):

$$k = |p|, \;\; F(t,p) = k^{-2}f(t,k)$$

the equation reads:

$$\frac{\partial f}{\partial t}(t,k) = \int_0^\infty b(k,k')q(f,f')dk', \quad f' = f(t,k')$$

$$q(f,f') = e^{-k}f(t,k')(k^2 + f(t,k)) - e^{-k'}f(t,k)(k'^2 + f(t,k'))$$

Equilibria-Conservation-Entropy

Regular Equilibria : $f_{\mu,0}(k) = \dfrac{k^2}{e^{k+\mu} - 1}, \;\; \mu \geq 0:$

Satisfy : $q(f_{\mu,0}, f'_{\mu,0}) \equiv 0$

The total number of particles:

$$N(f) = \int_0^\infty f(k)\, dk, \quad \frac{d}{dt} N(f(t)) = 0 \text{ along the trajectories.}$$

Remark No regular equilibria with mass larger than $N(f_{0,0})$:

$$N_\mu \equiv N(f_{\mu,0}) \leq N(f_{0,0}) = N_0$$

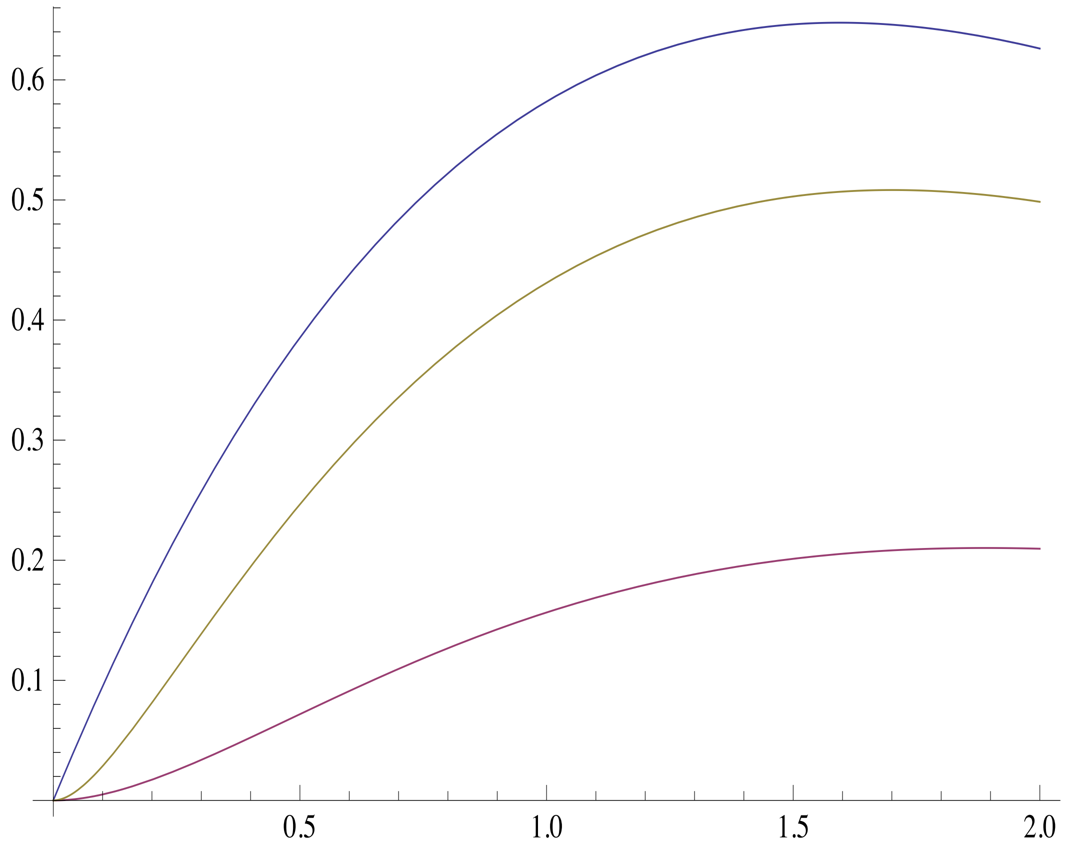

Figure 2: The regular equilibria $f_{\mu,0}$, $\mu = 0, 0.2, 1$.

BUT

The Boltzmann Compton equation has an entropy:

$$H(f) = \int_0^\infty h(f,k)\,dk, \quad \frac{d}{dt}H(f(t)) > 0$$
$$h(f,k) = (k^2+f)\log(k^2+f) - f\log f - k^2\log(k^2)$$

S. N. Bose & A. Einstein 20's works show:

If $N \le N_0 :\ \max_{N(f)=N} H(f) = H(f_{\mu,0})$ for some $\mu \ge 0$

If $N > N_0 :\ \max_{N(f)=N} H(f) = H(f_{0,\alpha})$ for some $\alpha \ge 0$

where $f_{0,\alpha} = \dfrac{k^2}{e^k - 1} + \alpha\delta(k) \equiv f_{0,0}(k) + \alpha\delta(k)$

Remark . If $b(k,k')$ is not continuous at $k' = 0$ sense of the equation for $f_{0,\alpha}$?

An interesting question

What happens near $k = 0$ as t increases ?

The Boltzmann Compton equation is simpler than the Boltzmann equation describing the boson-boson collisions in a dilute gas:

$$\begin{aligned}
\frac{\partial f}{\partial t}(t,p) &= Q(f)(t,p), \qquad t > 0,\ p \in \mathbb{R}^3. \\
Q(f)(t,p) &= \int\int\int_{\mathbb{R}^9} W\left(p,p_2,p_3,p_4\right) q(f) dp_2 dp_3 dp_4 \\
q\left(f\right) &= f_3 f_4 (1+f)(1+f_2) - f\, f_2 (1+f_3)(1+f_4) \\
W\left(p,p_2,p_3,p_4\right) &= \delta(p+p_2-p_3-p_4)\ \delta\left(|p|^2+|p_2|^2-|p_3|^2-|p_4|^2\right)
\end{aligned}$$

(Nordheim 1928)

Results on Boltz.-Compton

First results by E. Levich & V. Yakhot '77 & '78.... Later:

Theorem 1. *Under some conditions on b implying $b \in C([0,1) \times [0,1))$ assume:*

$$f_{in} \in \mathcal{E}_0 = \{h \in L^1(0,\infty), h \geq 0, \int_0^\infty (1+k)h(k)dk < \infty\}.$$

Then, there exists a unique global solution $f \in \mathbf{C}([0,\infty), \mathcal{E}_0)$ to Boltzmann Compton such that

$$\int_0^\infty f(k,t)dk = \int_0^\infty f_{in}(k)dk =: N \quad \forall t > 0.$$

Moreover:

(i) if $N = N_\mu : \lim_{t\to\infty}\int_0^\infty |f(k,t) - f_{\mu,0}(k)|dk = 0$

(ii) if $N > N_0 : f(\cdot, t) \rightharpoonup f_{0,0} + (N_0 - m)\delta$ *and:*

$$\lim_{t\to\infty}\int_{k>k_0} |f(k,t) - f_{0,0}(k)|dk = 0, \forall k_0 > 0.$$

Theorem 2. *Suppose that* $b \equiv 1$ *and*

$$f_{in}(k) \sim k \text{ as } k \to 0, \quad \textit{(for example)}$$
$$N(f_{in}) = N > N(f_{0,0}).$$

Then as $t \to +\infty$*, and uniformly on* $0 \leq kt \leq L$ *for any* $L > 0$ *fixed:*

$$F(t,k) - f_{0,0}(k) = \underbrace{(N - N_0)\, t\, \Phi(tk)}_{\to (N-N_0)\delta(k) \text{ as } t \to +\infty} (1 + o(1)) \quad \textit{as } t \to +\infty$$

where

$$\Phi(z) = N^2\, z\, e^{-N\, z}.$$

Theorem 1: E. & S. Mischler '01.

Theorem 2: E., S. Mischler & J.J.L.Velázquez. '04

But: Theorem 1 needs a "regular" kernel b.

Theorem 2 needs $b = 1$.

Singular kernel b

More realistic kernels $b(k,k')$ are rather singular near the origin.

Theorem 3. (M. Chane-Yook & A. Nouri '04) For some kernel b such that

$$b(k,k') \sim k^{-2}k'^{-2} \text{ as } k,k' \to 0$$

and initial data such that $(1+k^{-1})f_{in} \in L^1(\mathbb{R}^+)$ there exists a local (in time) weak solution of the equation such that

$$\frac{f(t,k)}{k} \in L^\infty((0,T),M^1(\mathbb{R}^+))$$

Remark. No result about global solutions, asymptotic behavior.

Simplified equations?

Kompaneets equation

Kompaneets in 1957: considers that the main contribution in the collision integral. comes from the region $|k' - k| << k$ and deduces the equation:

$$k^2 \frac{\partial g}{\partial t} = \frac{\partial}{\partial k} \left[k^4 \left(\frac{\partial g}{\partial k} + g + g^2 \right) \right], \ \left(g(t,k) = k^{-2} f(t,k) \right)$$

In order to have conservation of particles the boundary condition:

$$\lim_{k \to 0 \ or \ k \to +\infty} \left[k^4 \left(\frac{\partial g}{\partial k} + g + g^2 \right) \right] = 0$$

But for some solutions there is a positive time T for which this condition is not satisfied after T. (Some References: R. Caflisch & C. D. Levermore '86; E. ,Herrero & Velázquez '98; O. Kavian '02; N. Ben Abdallah, I. M. Gamba & G. Toscani '11; C. D. Levermore, H.Liu & R. Pego '11). More precisely:

Theorem. If $f_{in} \in \mathbf{C}_b(0,\infty)$, $\lim_{k\to\infty} k^4 f_{in}(k) = 0$. Then there exists a unique classical global solution such that $f(k,0) = f_{in}(k)$ such that, for some $C > 0$:

$$f(k,t) \leq \frac{C}{k^2} \quad \forall k \in (0,1), \forall t > 0; \quad \lim_{k\to\infty} k^4 (\frac{\partial f}{\partial k} + f + f^2) = 0.$$

Moreover, there are $f_{in} \in \mathbf{C}_b(0,\infty)$ such that for some $T^* > 0, C_1 > 0, C_2 > 0$:

- $\lim_{k\to 0} k^4 (\frac{\partial f}{\partial k} + f + f^2) = 0 \quad \forall t \in (0, T^*)$
- $\lim_{k\to 0} k^4 (\frac{\partial f}{\partial k} + f + f^2) > 0 \quad \forall t \geq T^*$

- $$\lim_{t\to T^*} (T^* - t) f(z(T^* - t), t) = \begin{cases} \frac{C_1}{z^2} & \text{if} \quad z > C_2 \\ 0 & \text{if} \quad 0 < z < C_2 \end{cases}$$

uniformly on compact subsets of (C_2, ∞) and $(0, C_2)$

Zel'dovich approximation

When $f >> k^2$ approximate: $(k^2 + f) \to f$ in the equation:

$$\frac{\partial f}{\partial t}(t,k) = f(t,k)\int_0^\infty f(t,k')(e^{-k} - e^{-k'})b(k,k')dk'$$

(Zel'dovich 1968). A more "classic" equation.

This approximation is very formal.

Similar "approximation" is used in other examples. For example by several authors (D. V. Semikoz et al.'74; Y. Kagan et al.'92; Y. Pomeau et al.'01) in the study of the Boltzmann equation for weakly interacting Bose particles.

Does it really make sense?

Use Theorem 2 to estimate the different terms for the case $b = 1$.

As $t \to +\infty$ and for $0 < k < \dfrac{L}{t}$:

$$(i) \quad f_t \sim kt\Phi'(kt) + \Phi; \qquad (ii) \quad \int_0^\infty dk' f' f e^{-k} \sim m_1 t\Phi(kt) e^{-k}$$

$$(iii) \quad \int_0^\infty dk' f' f e^{-k'} \sim t\Phi(kt) \int_0^\infty dk' f' e^{-k'} \text{ as } t \to +\infty:$$

$$\longrightarrow t\Phi(kt)\left(\int_0^\infty f_0 e^{-k} dk + (m_1 - m_0)\right)$$

$$(iv) \quad \int_0^\infty dk' f' k^2 e^{-k} \sim m_1 k^2 e^{-k}; \quad (v) \quad f(t,k) \int_0^\infty dk' k'^2 e^{-k'} \sim 2t\Phi(kt)$$

- (v) is of the same order than (ii) and (iii) (since the integral in that term converges to a positive constant as $t \to +\infty$).
- Linear terms are <u>not</u> asymptotically negligible compared to the nonlinear terms.

Does Zeldovich's approximation make sense?

We are led to study the equation:

$$\frac{\partial f}{\partial t}(t,k) = f(t,k)\int_0^\infty f(t,k')K(k,k')dk'$$
$$K(k,k') = (e^{-k} - e^{-k'})b(k,k')$$

More generally consider kernels $K(k,k')$ such that:

$$K(k,k') = -K(k',k)$$
$$K(k,k') < 0 \text{ if } k' < k; \;\; K(k,k') = 0 \Longleftrightarrow k = k'$$
$$K \in L^\infty(\mathbb{R}^+ \times \mathbb{R}^+) \;\text{ or }\; \sup_{k>0}\int_0^\infty |K(k,k')|dk' < +\infty.$$

Results II

Theorem 1. *For any* $f_0 \geq 0$ *in* $L^1(\mathbb{R}^+)$ *there exists a solution* $f \geq 0$ *in* $C([0,+\infty);L^1(\mathbb{R}^+))$. *It satisfies:*

$$\int_0^\infty f(t,k)\,dk = \int_0^\infty f(0,k)\,dk$$
$$supp(f(t)) = supp(f(0))$$

Theorem 2. *The set of stationary solutions of the equation is the set of Dirac measures* $\delta(k-a)$ *for all* $a \geq 0$.

Theorem 3. *If* f *is a solution with initial data* f_0 *with total mass* N *then*

$$f(t) \rightharpoonup N\,\delta(k-a), \text{ as } t \to +\infty; \quad a = \inf\{k;\ k \in supp(f_0)\}$$

Basic tools

- Multiply the equation by any function $\psi \in L^1 \cap L^\infty$ we obtain:

$$\frac{d}{dt}\int_0^\infty f(t,k)\psi(k)dk = \int_0^\infty \int_0^\infty f(t,k)f(t,k')K(k,k')\psi(k)dxdk'.$$

But since,

$$\int_0^\infty\int_0^\infty f(t,k)f(t,k)K(k,k')\psi(k)dkdk' = -\int_0^\infty\int_0^\infty f(t,k)f(t,k')K(k,k')\psi(k')dkdk',$$

we have,

$$\frac{d}{dt}\int_0^\infty f(t,k)\psi(k)dk = \frac{1}{2}\int_0^\infty\int_0^\infty f(t,k)f(t,k')K(k,k')(\psi(k)-\psi(k'))dkdk'.$$

Taking $\psi \equiv 1$, we have:

$$\frac{d}{dt}\int_0^\infty f(t,k)dk = 0$$

Taking $\psi = \mathbf{1}_{k>a}1$:

$$\frac{d}{dt}\int_a^\infty f(t,k)dk = \frac{1}{2}\int_a^\infty \int_0^a f(t,k)f(t,k')K(k,k')dk'dk - \frac{1}{2}\int_0^a \int_a^\infty f(t,k)f(t,k')K(x,y)dk'dk.$$

In particular if a belongs to the interior of the support of f_0,

$$\frac{d}{dt}\int_a^\infty f(t,k) < 0, \quad \forall t > 0.$$

(If a is in the interior of the support of f_0, it is in the interior of the support of $f(t)$ for all $t > 0$).

- The reduced equation has an entropy:

$$H(t) = \int_0^\infty k f(t,k)\,dk, \quad \frac{dH}{dt} = D(f)(t)$$
$$D(f)(t) = \frac{1}{2}\int_0^\infty \int_0^\infty f(t,k) f(t,k') K(k,k')(k-k')\,dk\,dk' < 0$$

- If, for any measure F on $\mathbb{R}^+$ we denote

$$\ell(F) = \min\{k \geq 0; k \in supp(F)\}$$

then the following holds:

Theorem 4. *For any non negative measure F such that*

$$\int_0^\infty k dF(k) < +\infty$$

the following are equivalent:

1.- $F = m\,\delta(k-a)$ for $a \geq 0$.

2.- F is the solution of the minimisation problem:

$$H(F) = \min_{M(F')=m, \ell(F')=a} H(F')$$

3.- $D(F) = 0$ and $\ell(F) = a$.

4.- $F \displaystyle\int_0^\infty K(k,k')dF(k') = 0$ and $F \geq 0$.

Proof of $1 \iff 2$. $1 \implies 2$: Notice first that for any non negative measure F whose mass is m and $\ell(F) = a$:

$$\int_0^\infty k\, F(k)\, dk = \int_a^\infty k\, F(k)\, dk \geq a \int_0^\infty F(k)\, dk = a\, m.$$

Therefore

$$\inf_{M(F')=m, \ell(F')=a} H(F) \geq a\, m.$$

Moreover, if $F = m\delta(k-a)$, $H(F) = a\, m$, from where

$$\inf_{M(F')=m, \ell(F')=a} H(F) = \min_{M(F')=m, \ell(F')=a} H(F) = a\, m.$$

$2 \Longrightarrow 1.$ On the other hand, if F satisfies $\ell(F) = a$, $M(F) = m$ and

$$\int_a^\infty kF(k)\,dk = a\,m$$

then,

$$\int_a^\infty (k-a)F(k)\,dk = 0$$

from where

$$(k-a)F(k) \equiv 0, \quad a.e.\,x > 0$$

and therefore $F = m\,\delta(k-a)$.

Results III

Theorem 5. *Suppose $b \equiv 1$. If f is a solution with initial data f_0 such that*

$$\int_0^\infty f_0(k)dk = N$$

$$u_0(k) \sim k \quad \textit{as } k \to 0$$

then:

$$u(t,k) \sim N^3 t^2 k e^{-Ntk} \quad \textit{as} \quad t \to +\infty.$$

Sketch of the Proof. The equation may be written:

$$\frac{\partial f}{\partial t}(t,k) = N\, e^{-k}\, f(t,k) - \varepsilon(t)\, f(t,k)$$

$$\varepsilon(t) = \int_0^\infty e^{-k'}\, f(t,k')\, dk'.$$

Define:

$$h(t,k) = f(t,k)\,\exp\left(\int_0^t \varepsilon(s)\,ds\right),$$

we have

$$\frac{\partial h}{\partial t}(t,k) = N\,e^{-k}\,h(t,k) \Longrightarrow h(t,k) = f_0(k)\,e^{Ne^{-k}t}$$

$$f(t,k) = f_0(k)e^{Ne^{-k}t}\exp\left(-\int_0^t \varepsilon(s)\,ds\right)$$

In order to determine $f(t,k)$ we consider the auxiliary function

$$\lambda(t) = \varepsilon(t)\,\exp\left(\int_0^t \varepsilon(s)\,ds\right).$$

By the definition of ε one has:

$$\begin{aligned}
\lambda(t) &= \frac{d}{dt}\exp\left(\int_0^t \varepsilon(s)\,ds\right) \\
&= \int_0^\infty e^{-y}h(t,y)\,dy = \int_0^\infty e^{-y}f_0(y)\,e^{Ne^{-y}t}\,dy \qquad (1)
\end{aligned}$$

We just need the behavior of the right hand side of (1) as $t \to +\infty$. That only depends on the behavior of $f_0(k)$ as $k \to 0$. If, for example

$$f_0(k) = k + o(k) \text{ as } k \to 0$$

then

$$\int_0^\infty e^{-k'}f_0(k')\,e^{Ne^{-k'}t}\,dk' \quad = \quad e^{Nt}\left(\frac{1}{N^2t^2} + \mathcal{O}\left(\frac{1}{t^3}\right)\right), \text{ as } t \to +\infty.$$

Integration of this estimate between 0 and t gives:

$$\exp\left(\int_0^t \varepsilon(s)\,ds\right) = e^{Nt}\left(\frac{1}{N^3t^2} + \mathcal{O}\left(\frac{1}{t^3}\right)\right) + o(1), \text{ as } t \to +\infty.$$

Therefore, for all $k > 0$ and $t \to +\infty$:

$$\begin{aligned} f(t,k) &= h(t,k)\exp\left(-\int_0^t \varepsilon(s)\,ds\right) \\ &= \frac{f_0(k)\,e^{Ne^{-k}t}}{e^{Nt}\left(\frac{1}{N^3t^2} + \mathcal{O}\left(\frac{1}{t^3}\right)\right) + o(1)} \\ &= N^3\,t^2\,k\,e^{-Nkt}\,(1+\cdots) \\ &= N\,t\Phi(tk)\,(1+\cdots) \end{aligned}$$

From the two results on the long time behavior near $k = 0$ we deduce:

Corollary Let f and F be such that:

$$\frac{\partial f}{\partial t}(t,k) = \int_0^\infty \left(e^{-k} f(t,k')(k^2 + f(t,k)) - e^{-k'} f(t,k)(k'^2 + f(t,k'))\right) dk'$$

$$\frac{\partial F}{\partial t}(t,k) = F(t,k)\int_0^\infty F(t,k')(e^{-k} - e^{-k'})dk'$$

$$f(0,k) = F_0(k) = f_{in}(k), \quad N = \int_0^\infty f_{in}(x)dx > N_0$$

Then, $\forall L > 0:$ $\boxed{\lim_{t\to+\infty} \int_0^{L/t} |(f(t,k) - f_{0,0}(k)) - (N - N_0)F(t,k)|dk = 0}$

Proof We have shown:

$f(t,x) - f_{0,0}(x) = (N - N_0)t\Phi(tk)(1 + o(1))$ for $k \in (0, L/t)$ as $t \to +\infty$

$F(t,x) = t\Phi(tk)(1 + o(1))$ as $t \to +\infty$.

Remark 6. *Same scale as in Compton-Boltzmann. Not the same mass. The reduced equation may be writen:*

$$\begin{aligned} \frac{\partial F}{\partial t}(t,k) &= F(t,k)\int_0^\infty F(t,k')\left(e^{-k}-e^{-k'}\right)\,dk' \\ &= N\,e^{-k}\,F(t,k) - F(t,k)\int_0^\infty e^{-k'}\,F(t,k')\,dk'. \end{aligned}$$

The Boltzmann-Compton equation may be writen:

$$f(t,k) = f_{0,0} + v(t,k), \quad \int_0^\infty v(t,k)dk = N - N_0$$

$$\frac{\partial v}{\partial t}(t,k) = \boxed{(N-N_0)e^{-k}v(t,k) - v(t,k)\int_0^\infty v(t,k')e^{-k'}dk'} -$$

$$\underbrace{-N_0(1-e^{-k})v(t,k) - f_{0,0}(k)\left(\int_0^\infty v(t,k')e^{-k'}dk' + (N-N_0)\right)}_{\text{smaller as } k\to 0}$$

$$\boxed{\frac{\partial v}{\partial t}(t,k) = (N-N_0)e^{-k}v(t,k) - v(t,k)\int_0^\infty v(t,k')e^{-k'}dk'} -$$

$$-N_0(1-e^{-k})v(t,k) - f_{0,0}(k)\left(\int_0^\infty v(t,k')e^{-k'}dk' + (N-N_0)\right)$$

The solution to the boxed equation behaves like:

$$(N-N_0)^3\, t^2\, k\, e^{-(N-N_0)kt}\,(1+\cdots)\,,\ \textit{as}\ t \to +\infty$$

But we must instead write the complete Boltzmann Compton equation:

$$\frac{\partial v}{\partial t}(t,k) = Nv(t,x)(e^{-k}-1) + (v+f_{0,0})\int_0^\infty (1-e^{-k'})v(t,k')dk$$

whose solution behaves like:

$$(N-N_0)N^2\, t^2\, k\, e^{-Nkt}\,(1+\cdots)\,,\ \textit{as}\ t \to +\infty$$

Singular kernels

A reasonable estimate of a more physical kernel gives:

$$b(k,k') = \frac{e^{\beta\frac{k+k'}{2}}}{k^2k'^2}\int_{(k-k')^2}^{(k+k')^2} \frac{e^{-\beta\frac{|k-k'|^2}{2}-\frac{\beta s}{8}}}{\sqrt{s}}ds$$

for some $\beta > 0$. As $k \to 0$ and $k' \to 0$ we have:

$$b(k,k') \sim C\frac{\min(k,k')}{k^2k'^2}$$

For the sake of simplicity we consider kernels of the form $b(k,k') = (kk')^{-\alpha}$.

We can prove: existence of a local classical solution. Problem: global or not?

Conservation laws

A regular type of kernel may be as follows (Zel'dovich, 68):

$$G_a(k) = (4a\pi)^{-1/2} e^{-\frac{|k|^2}{4a}}$$

$$K_a(k,k') = \frac{\partial G_a}{\partial k}(k-k') = (4a\pi)^{-1/2} \frac{(k'-k)}{2a} e^{-\frac{|k-k'|^2}{4a}}$$

All what was said for good kernels applies to K_a. For every $a > 0$ we can then solve the problem:

$$\frac{\partial f}{\partial t}(t,k) = f(t,k) \int_0^\infty K_a(k,k')\, f(t,k')\, dk' = f(t,k) \frac{\partial}{\partial k} \left(G_a * f\right)(t,k).$$

It is formally clear that as $a \to 0$ we obtain in the limit the conservation law:

$$\frac{\partial f}{\partial t}(t,k) = f(t,k) \frac{\partial f}{\partial k}(t,k) \quad \text{but} \cdots$$

Initial data in L^∞.

Theorem 7. *Suppose that K satisfies:*

$$\kappa = \sup_{k>0} \int_0^\infty |K(k,k')|dk' < +\infty$$

Then, for any initial data $f_0 \in L^\infty(\mathbb{R}^+)$, $f_0 \geq 0$ there exists a $T^* > 0$ such that there is a unique non negative solution $f \in C([0,T^*); L^\infty(\mathbb{R}^+))$ of the equation. Moreover, for every $t \in [0,T^*)$:

$$||f(t)||_{L^\infty(\mathbb{R}^+)} \leq ||f_0||_\infty e^{\kappa \int_0^t ||f(s)||_\infty ds}.$$

Finally, either $T^* = +\infty$ or $\lim_{t\to T^*} ||f(t)||_\infty = +\infty$.

Question: global or blow up in finite time?