BOSE CONDENSATES IN INTERACTION WITH EXCITATIONS

A KINETIC MODEL

Leif Arkeryd, *Chalmers*, *Sweden*,
Anne Nouri, *Aix Marseille University*, *France*.

This talk deals with mathematical questions for Bose gases below the temperature T_{BEC} where Bose-Einstein condensation sets in.

The model considered is of two-component type, consisting of :

- a kinetic equation for the distribution function of a gas of (quasi-)particles interacting with a Bose condensate,

- a Gross-Pitaevskii equation describing the condensate.

Denote by, n_c the non-equilibrium density of the atoms in the condensate, m the atomic mass, g a scattering length, and

$$E_p = \sqrt{\frac{p^4}{4m^2} + \frac{g n_c}{m} p^2}$$

the excitation energy.

, the kinetic equation for the distribution function of a gas of (quasi-)particles interacting with a Bose condensate is in an isotropic setting,

$$\frac{\partial f}{\partial t} = C(f, n_c).$$

The collision term is

$$C(f, n_c)(p) = n_c \int \bar{B}\,\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3))dp_1\,dp_2\,dp_3.$$

Let $p_0 = \sqrt{2mgn_c}$ be a characteristic momentum.

The kernel $\bar{B}$ is bounded by a multiple of

$$B := \Big(\frac{|p_1|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_2|}{\sqrt{n_c}} \wedge 1\Big)\Big(\frac{|p_3|}{\sqrt{n_c}} \wedge 1\Big),$$

in the physically interesting cases when asymptotically

- either all $|p_i| << p_0$,
- or all $|p_i| >> p_0$,
- or one $|p_i| << p_0$ and the others $>> p_0$.

The three cases are relevant for low (resp. intermediate temperatures compared to T_{BEC}, resp. collisions of low temperature phonons with high temperature excitations).

The Bose condensate density $n_c(t)$ satisfies

$$n_c'(t) = -\int C(f, n_c)\, dp.$$

The main result.

Theorem

Let $n_{ci} > 0$ and $f_i(p) = f_i(|p|) \in L^1_+$ be given with $f_i(p)|p|^{2+\gamma} \in L^1$ for some $\gamma > 0$.
There exists a nonnegative solution

$$(f, n_c) \in C^1([0,\infty); L^1_+) \times C^1([0,\infty))$$

to the initial value problem.
The condensate density n_c is locally bounded away from zero for $t > 0$.
The excitation density f conserves momentum and has energy locally bounded in time.
Total mass $M_0 = n_{ci} + \int f_i(p)dp$ is conserved, and the moment $\int |p|^{2+\gamma} f dp$ is locally bounded in time.

In the low temperature case, if the mathematical condition corresponding to the physics requirement $|p| << p_0$ is taken as $|p| \leq p_0^2 := \lambda$, the proof of the main theorem simplifies. It holds that

Theorem

Let $n_{ci} > 0$ and $f_i(p) = f_i(|p|) \in L^1_+$ be given.
There exists a nonnegative solution

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In the intermediate temperature case, without the cut-off function χ in the collision operator, mention the existence result [N 2005, 'Bose-Einstein condensates at very low temperatures...'] considering the excitation density f in measure sense.
With the cut-off function χ included, existence also holds in the present L^1-setting.

Theorem

Let $n_{ci} > 0$ *and* $f_i(p) = f_i(|p|) \in L^1_+$ *with* $f_i(p)|p|^{2+\gamma} \in L^1$ *for some* $\gamma > 0$.

There exists a nonnegative solution

$$(f, n_c) \in C^1([0,\infty); L^1_+) \times C^1([0,\infty))$$

to the initial value problem.

The condensate density n_c *is locally bounded away from zero for* $t > 0$.

The excitation density f *conserves momentum.*

Total mass is conserved, as well as the energy type integral

$$\int \frac{p^2}{2m} f_i(p)dp + \frac{1}{2} g n_{ci} \int f_i(p)dp + gM_0(\int f_i(p)dp + \frac{1}{2} n_c).$$

The moment $\int |p|^{2+\gamma} f dp$ *is locally bounded in time.*

Proof of the main theorem.

The collision operator is

$$C(f,n_c)(p) = n_c \int B\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1)$$
$$-\delta(p - p_2) - \delta(p - p_3)]((1 + f_1)f_2 f_3 - f_1(1 + f_2)(1 + f_3))dp_1 dp_2 dp_3.$$

First,

$$\int \varphi(p) C(f,n_c)(p)dp$$
$$= n_c \int B(\varphi(p_1) - \varphi(p_2) - \varphi(p_3))\delta(p_1 - p_2 - p_3)$$
$$\delta(E_1 - E_2 - E_3)(f_2 f_3 - f_1(1 + f_2 + f_3))dp_1 dp_2 dp_3.$$

Hence $n_c(t) + \int f(t,|p|)dp = M_0$, i.e. total mass is conserved.

The energy (resp. the condensate density) is bounded from above (resp. from below) locally in time as follows.

Lemma

Let the initial data (f_i, n_{ci}) *satisfy*

$$0 < n_{ci} < M_0, \quad n_{ci} + \int f_i(|p|)dp = M_0.$$

Then there is $T_0 > 0$ *such that*

$$\frac{n_{ci}}{2} \leq n_c(t), \quad \int E(p, n_c)f(t,p)dp < k, \quad t \in [0, T_0],$$

for any nonnegative solution (f, n_c) *to the system.*

Proof of the lemma.

For any nonnegative solution (f, n_c) to the system,

$$|\frac{n_c'}{n_c}(t)| \leq \int (\frac{|p_1|}{\sqrt{n_c}} \wedge 1)(\frac{|p_2|}{\sqrt{n_c}} \wedge 1)(\frac{|p_3|}{\sqrt{n_c}} \wedge 1)\delta(p_1 - p_2 - p_3)$$
$$\delta(E_1 - E_2 - E_3)(f_2 f_3 + f_1(2f_2 + 1))dp_1 dp_2 dp_3$$
$$=: X_1 + X_2 + X_3.$$

Using spherical coordinates for p_2 and p_3, with axis directed by p_2 and azimuthal angle φ_3 for p_3, setting $|p| = r$, and performing the change of variables $\varphi_3 \to s = cos\varphi_3$,

$$X_1 \leq 2k \int_0^\infty r_2^2(\frac{r_2}{\sqrt{n_c}} \wedge 1) f(t, r_2) \int_0^{r_2} r_3^2(\frac{r_3}{\sqrt{n_c}} \wedge 1) f(t, r_3) Y_1 dr_3 dr_2,$$

where

$$Y_1 = \int_{-1}^1 \delta(F_1(s))ds,$$

$$F_1(s) := \sqrt{(r_2^2 + r_3^2 + 2r_2r_3s)^2 + n_c(r_2^2 + r_3^2 + 2r_2r_3s)} - S_1,$$

$$S_1 := \sqrt{r_2^4 + n_c r_2^2} + \sqrt{r_3^4 + n_c r_3^2}.$$

F_1 vanishes for a single value s_1 of s. Then,

$$Y_1 \leq \frac{1}{r_2 r_3}.$$

Hence,

$$X_1 \leq 2k \int_0^\infty r_2 \Big(\frac{r_2}{\sqrt{n_c}} \wedge 1\Big) f(t, r_2) \int_0^{r_2} r_3 \Big(\frac{r_3}{\sqrt{n_c}} \wedge 1\Big) f(t, r_3) dr_3 dr_2$$

$$\leq \frac{2k}{n_c} \Big(\int f(t,p) dp \Big)^2.$$

Similarly,

$$X_2 \leq \frac{2k}{n_c}\Big(\int f(t,p)dp\Big)^2.$$

Finally,

$$X_3 \leq k\sqrt{n_c}\int f(t,p)dp$$
$$+\frac{k}{2\sqrt{n_c}}\int p^2 f(t,p)dp \leq M_0 k\sqrt{n_c} + \frac{k}{2\sqrt{n_c}}\int p^2 f(t,p)dp.$$

And so,

$$|n_c'(t)| \leq k(4M_0^2 + M_0 n_c^{\frac{3}{2}} + \sqrt{n_c}\int p^2 f(t,p)dp).$$

Denote by $G(t,n) = \int E(p,n)f(t,p)dp$. Then

$$\frac{\partial G}{\partial n} = \int \frac{|p|}{2\sqrt{p^2+n}} f(t,p)dp \in [0, \frac{M_0}{2}].$$

Hence,

$$G(t, n_c(t)) \leq G(t, n_{ci}) + M_0^2.$$

Moreover,

$$\frac{d}{dt}G(t,n_{ci}) = \int |p|\sqrt{p^2+n_{ci}}C(f,n_c)(t,p)dp$$

$$= n_c \int |\bar{A}|^2 \Big(|p_1|(\sqrt{p_1^2+n_{ci}} - \sqrt{p_1^2+n_c(t)})$$

$$-|p_2|(\sqrt{p_2^2+n_{ci}} - \sqrt{p_2^2+n_c(t)})$$

$$-|p_3|(\sqrt{p_3^2+n_{ci}} - \sqrt{p_3^2+n_c(t)})\Big)$$

$$\delta(p_1-p_2-p_3)\delta(|p_1|\sqrt{p_1^2+n_c(t)} - |p_2|\sqrt{p_2^2+n_c(t)}$$

$$-|p_3|\sqrt{p_3^2+n_c(t)})(f_2f_3 - f_1(1+f_2+f_3)dp_1dp_2dp_3.$$

It follows from

$$|p|\,|\sqrt{p^2+n_{ci}}-\sqrt{p^2+n_c(t)}\,|\leq|\,n_{ci}-n_c(t)\,|\leq M_0,\quad p\in R^3,$$

and similar computations as in the previous control of X_1, X_2 and X_3, that

$$|\frac{d}{dt}G(t,n_{ci})|\leq 2kM_0n_c(t)\Big(\frac{4M_0^2}{n_c(t)}+2M_0\sqrt{n_c(t)}+\frac{1}{\sqrt{n_c}}\int p^2f(t,p)dp\Big)$$
$$\leq 2kM_0n_c(t)\Big(\frac{4M_0^2}{n_c(t)}+2M_0\sqrt{n_c(t)}+\frac{G(t,n_{ci})}{\sqrt{n_c}}\Big).$$

And so,

$$G(t,n_c(t))\leq M_0^2+G(0,n_{ci})\exp{(2M_0^{\frac{3}{2}}t)}$$
$$+k(8M_0^3+2M_0^4+2M_0^{\frac{5}{2}})(\exp{(2M_0^{\frac{3}{2}}t)}-1).$$

The lemma follows.

If the solution exists on $[0, T[$, then it follows from a refinement of the previous proof that $\inf_{[0,T[} n_c(t) > 0$. Indeed, one proves that for some $k > 0$,

$$n_c'(t) \geq k - 2M_0 n_c^{\frac{3}{2}} - \sqrt{n_c} \int p^2 f(t,p)dp, \quad t \in [0, T[,$$

with the integral $\int p^2 f(t,p)dp$ bounded.

Lemma

For any nonnegative (f, n),

$$|\int C(f,n)(p)dp| \leq k\Big(\int f(p)dp\Big)^2 + k\sqrt{n}\int p^2 f(p)dp.$$

Lemma

Given $0 < n_* < M_0$, *there is a constant* k *such that for any* $n \in [n_*, M_0]$ *and isotropic functions* $(f, g) \in L^1_+(\mathbb{R}^3) \times L^1_+(\mathbb{R}^3)$ *with* L^1 *norm bounded by* M_0,

$$\int |(C(f,n) - C(g,n))(p)|dp \leq k\int (1 + \sqrt{n}p^2)|(f-g)(p)|dp.$$

Lemma

For any $\gamma \in [0, 1]$,

$$\int |p|^{2+2\gamma} f(t,p)dp \le \int |p|^{2+2\gamma} f_i(p)dp + M_0 t \sup_s \left(\int (1+p^2) f(s,p)dp \right)$$

Proof.

Multiply the equation satisfied by f by $|p|^{2+2\gamma}$ and integrate it on $(0, t) \times \mathbb{R}^3$, so that

$$\int |p|^{2+2\gamma} f(t,p)dp + \int_0^t n_c(s) \int B\big(|p_1|^{2+2\gamma} - |p_2|^{2+2\gamma} - |p_3|^{2+2\gamma}\big) f_1 \delta(p_1 = p_2 + p_3)\delta(E_1 = E_2 + E_3) dp_{123} ds$$

$$= \int |p|^{2+2\gamma} f_i(p)dp + \int_0^t n_c(s) \int B\big(|p_1|^{2+2\gamma} - |p_2|^{2+2\gamma} - |p_3|^{2+2\gamma}\big) (f_2 f_3 - f_1(f_2 + f_3))\delta(p_1 = p_2 + p_3)\delta(E_1 = E_2 + E_3) dp_{123} ds.$$

It is sufficient to prove that there is a positive constant $\tilde{K}$ such that

$$0 \leq r_1^{2+2\gamma} - r_2^{2+2\gamma} - r_3^{2+2\gamma} \leq \tilde{K}(1 + r_2^2)(1 + r_3^2), \qquad (2)$$

when $E_1 = E_2 + E_3$. Indeed, the second term in the left member of the previous slide will then be nonnegative, whereas the second term in the right member will be bounded from above by $M_0 t \sup_{s \leq t} (\int (1 + p^2) f(s,p) dp)^2$. Since

$$r_i^2 = \frac{\sqrt{n^2 + 4E_i^2} - n}{2}, \quad 1 \leq i \leq 3,$$

(2) holds if there is a positive constant K such that

$$0 \leq \left(\sqrt{n^2 + 4(E_2 + E_3)^2} - n\right)^{1+\gamma} - \left(\sqrt{n^2 + 4E_2^2} - n\right)^{1+\gamma} - \left(\sqrt{n^2 + 4E_3^2} - n\right)^{1+\gamma}$$

$$\leq K\left(\sqrt{n^2 + 4E_2^2} - n + 2\right)\left(\sqrt{n^2 + 4E_3^2} - n + 2\right), \quad (E_2, E_3) \in (\mathbb{R}_+)^2.$$

Construction of solutions on a small interval of time

For any initial data f_i and $n_{ci} > 0$, there is a positive time T_0 and $n_* > 0$ such that any solution (f, n_c) of the initial value problem is such that

$$n_c(t) \geq n_*, \quad t \in [0, T_0].$$

Let

$$\mathcal{K} := \{n \in C([0, T_0]); n(0) = n_{ci}, \quad \frac{n_*}{2} \leq n(t) \leq M_0, \quad t \in [0, T_0]\}.$$

A local in time solution to the problem is found as a fixed point of the following map.

Let a (large) truncation value P be defined for the linear part of the collision operator. Let Φ be the map defined on $\mathcal{K}$ by $\Phi(n) = m$, where

$$m(t) = M_0 - \int f(t,p)dp, \quad t \in [0, T_0],$$

and f is the mild solution in $C([0,\tau_0]; L^1)$ for some τ_0 defined below with $0 < \tau_0 \leq T_0$, to

$$\frac{\partial f}{\partial t} = C^P(f,n), \quad f(0,p) = f_i(p), \tag{5}$$

with

$$\begin{aligned} C^P(f,n) = n \int & B\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3)[\delta(p - p_1) \\ & -\delta(p - p_2) - \delta(p - p_3)](f_2 f_3 - f_1(f_2 + f_3))dp_1 dp_2 dp_3 \\ & -n \int \chi_{|p|<P} B\delta(p_1 - p_2 - p_3)\delta(E_1 - E_2 - E_3) \\ & (\delta(p - p_1) - \delta(p - p_2) - \delta(p - p_3)) f_1 dp_1 dp_2 dp_3. \end{aligned}$$

Here $\chi_{|p|<P}$ is the characteristic function of the set where $|p| < P$.

Writing the equation satisfied by f in exponential form and estimating the solution from below by the term containing the initial value, it follows that the bound of n from below, n_*, can be taken independent of P.
Given n, a mild solution f for (5) can be constructed as the limit of the nonnegative sequence (f_j), defined by $f_0 = f_i$ and

$$\frac{\partial f_{j+1}}{\partial t} + f_{j+1} C_l^P(f_j, n) = C_g^P(f_j, n),$$
$$f_{j+1}(0, p) = f_i(p).$$

The collision frequency is

$$\begin{aligned} C_l^P(f, n) = n[2 \int B\,\delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3) f_2 dp_2 dp_3 \\ +2 \int B\,\delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p) f_2 dp_1 dp_2 \\ + \int \chi_{|p|<P} B\,\delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3) dp_2 dp_3], \end{aligned}$$

which preserves positivity together with the gain term $C_g^P(f, n) = C^P(f, n) + f C_l^P(f, n)$.

For any nonnegative functions $f, g \in L^1$ and any $n \in [0, M_0]$, it holds that

$$\int |(C^P(f,n) - C^P(g,n))(p)|dp \leq kn(P^2 + \int f(p)dp + \int g(p)dp) \int |(f-g)(p)|dp. \quad (6$$

For $\tau_0 > 0$ smaller than $\frac{c}{\int f_i(p)dp + P^2}$, where c is a suitable constant, the sequence (f_j) is uniformly bounded by $2\int f_i(p)dp$ and converges in $C([0,\tau_0]; L^1)$ to a mild solution f of (5) (using (6) and induction), since

$$\sup_{t\in[0,\tau_0]} |(f_{j+1} - f_j)(t,\cdot)|_{L^1} \leq k\tau_0 \sup_{t\in[0,\tau_0]} |(f_j - f_{j-1})(t,\cdot)|_{L^1}, \quad j \in \mathbb{N}.$$

The nonnegative solution f is unique in $C([0,\tau_0]; L^1)$ by the L^1-Lipschitz property of $C^P(\cdot, n)$. The time-interval $[0,\tau_0]$ can be so chosen that $m(t) = \Phi(n)(t) \geq \frac{1}{2}n_*$ uniformly for $n \in \mathcal{K}$ and $0 \leq t \leq \tau_0$.

The map Φ is continuous. Indeed, let $(n, \tilde{n}) \in \mathcal{K} \times \mathcal{K}$ and $m = \Phi(n)$ resp. $\tilde{m} = \Phi(\tilde{n})$. Then for $t \leq \tau_0$,

$$\int |(f - \tilde{f})(t,p)|dp \leq kt \parallel n - \tilde{n} \parallel_\infty | + k \int_0^t \int |(f - \tilde{f})(s,p)|dpds.$$

Consequently, for τ small

$$\sup_{t \in [0,\tau]} \int |(f - \tilde{f})(t,p)|dp \leq k\tau \parallel n - \tilde{n} \parallel_\infty, \tag{7}$$

and so

$$\parallel m - \tilde{m} \parallel_\infty \leq k\tau \parallel n - \tilde{n} \parallel_\infty .$$

The continuity of Φ on $[0, \tau_0]$ follows. Moreover, the map Φ is compact by Arzela-Ascoli. Indeed $\Phi(\mathcal{K})$ is bounded on $[0, \tau_0]$, since $\frac{1}{2}n_* \leq \Phi(n)(t) \leq M_0$ for $t \in [0, \tau_0]$, $n \in \mathcal{K}$.

Besides the map Φ is equicontinuous, since

$$\begin{aligned}|\Phi(n)(t_1) - \Phi(n)(t_2)| &= |\int f(t_1, p)dp - \int f(t_2, p)dp| \\ &\leq |t_1 - t_2| \sup_{t\in[0,\tau_0]} \int |C(f, n)(t, p)|dp \\ \leq k\Big(\sup_{t\in[0,\tau_0]} \int f(t, p)dp(P^2 + \sup_{t\in[0,\tau_0]} \int f(t, p)dp)\Big)|t_1 - t_2| \\ \leq k\Big(2 \int f_i(p)dp(P^2 + 2 \int f_i(p)dp)\Big)|t_1 - t_2|, \quad n \in K.\end{aligned}$$

Consequently, there is a pair of nonnegative functions

$$(f^P, n_c^P) \in C([0, \tau_0], L^1) \times C([0, \tau_0]),$$

satisfying

$$\frac{\partial f^P}{\partial t} = C^P(f^P, n_c^P), \quad \frac{dn_c^P}{dt} = -\int C^P(f^P, n_c^P)dp \qquad (8)$$
$$f(0, p) = f_i(p), \quad n_c^P(0) = n_{ci},$$

in mild form with a truncation for $| p |> P$ in the linear part of the collision operator.
Since $\int C^P(f^P, n_c^P)dp$ is continuous in t, the solution n_c^P is continuously differentiable in t and satisfies (8) in strong form.

Lemma

The family $(C^P(f^P, n^P)(t))$, $t \in [0, T_0]$, *with values in* L^1, *is* t*-continuous in the* L^1*-norm, uniformly with respect to* P *and* t*.*

The conservation of total mass follows from the fixed point property.
The boundedness of the energy of f^P is similar to the previous formal proof.
The integrals $\int (1 + p^2) f^P dp$ are also uniformly in P bounded.
Observing that n_* is so chosen that for any P, $n_c^P \geq n_*$ on any subinterval $[0, T_0']$ of $[0, T_0]$ where n_c^P exists, the result can for each P be extended by iteration to the whole interval of time $[0, T_0]$.

The f^P's are also the limits in $C([0, T_0]; L^1_{1+p^2})$ of increasing sequences $(\tilde{f}_j^P)$, defined by $\tilde{f}_0^P = 0$ and

$$\frac{\partial \tilde{f}_{j+1}^P}{\partial t} + \tilde{f}_{j+1}^P \Big(n_c^P \int B \chi_{|p|<P} \delta(p - p_2 - p_3) \delta(E_p - E_2 - E_3) dp_2 dp_3 + \tilde{C}_l(f^P, n_c^P) \Big) = \tilde{C}_g(\tilde{f}_j^P, n_c^P)$$
$$+ 2n_c^P \int B \chi_{|p|<P} \delta(p_1 - p_2 - p) \delta(E_1 - E_2 - E_p) f_1^P dp_1 dp_2,$$
$$\tilde{f}_{j+1}^P(0, p) = f_i(p) \chi_{|p|<P}.$$

Here $\tilde{C}_l$ is defined by

$$\tilde{C}_l(f, n) = 2n[\int B \delta(p - p_2 - p_3) \delta(E_p - E_2 - E_3) f_2 dp_2 dp_3 + \int B \delta(p_1 - p_2 - p) \delta(E_1 - E_2 - E_p) f_2 dp_1 dp_2],$$

and

$$\tilde{C}_g(f,n) = n[\int B\delta(p-p_2-p_3)\delta(E_p-E_2-E_3)f_2f_3dp_2dp_3$$
$$+2\int B\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1f_2dp_1dp_2$$
$$+2f\int B\delta(p_1-p_2-p)\delta(E_1-E_2-E_p)f_1dp_1dp_2].$$

It will be used in the study of $lim_{P\to\infty} f^P$ below, that such $\tilde{f}_j^P$'s share with the f^P's any uniform bound for $(2+\gamma)$-moments. One proves by induction that $(\tilde{f}_j^P)$, $j\in\mathbb{N}$, is an increasing sequence of nonnegative functions bounded by f^P.

It remains to prove that a subsequence of (f^P, n_c^P) converges in $C([0, T_0], L^1_{1+p^2}) \times C([0, T_0])$, and that its limit is solution to the problem.
Using Arzela-Ascoli, the sequence (n^P) is compact in $C([0, T_0])$.

Lemma

Given $t \in [0, T_0]$, the family $(g^P(t)) := (\int B\chi_{|p|<P}\delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p)f_1^P(t)dp_1 dp_2)$ is compact in L^1. This also holds for the family $(\tilde{C}_l(f^P, n_c^P)(t))$.

Proof.
First,

$$\int_{|p|>K} g^P(p)dp \leq k \int_K^{+\infty} r_1 f^P(r_1) \int_K^{r_1} r_3 dr_3 dr_1$$
$$\leq k \int_K^{+\infty} r_1^3 f^P(r_1)dr_1 \leq \frac{k}{K} \int p^2 f^P(p)dp,$$

which uniformly in P and t tends to zero when $K \to +\infty$.

One then proves that for fixed $K > 0$,

$$\lim_{h \to 0} \int |g^P(p+h)\chi_{|p+h|<K} - g^P(p)\chi_{|p|<K}|dp = 0,$$

uniformly with respect to P and t.
For the family $\tilde{C}_l(f^P, n_c^P)$ it is enough to consider a sequence (P_j) tending to infinity, for which $(n_c^{P_j})$ is uniformly in t convergent. From there the proof is similar to the previous case.

We can now take a subsequence $(g^{P_l}, \tilde{C}_l(f^{P_l}, n_c^{P_l}), n_c^{P_l})$ with P_l tending to infinity, converging for rational t to a limit (g, h, n_c), and will for such t prove that (f^{P_l}) is a Cauchy sequence in L^1. The Cauchy property for irrational t then follows from a former lemma.

To prove that (f^{P_l}) is a Cauchy sequence in L^1, split $f^{P_{l'}} - f^{P_{l''}}$ into

$$f^{P_{l'}} - f^{P_{l''}} = (f^{P_{l'}} - \tilde{f}_j^{P_{l'}}) + (\tilde{f}_j^{P_{l'}} - \tilde{f}_j^{P_{l''}}) + (\tilde{f}_j^{P_{l''}} - f^{P_{l''}}).$$

It is enough to prove the convergence on compact p-sets. From the cancelation of the inhomogeneous term in the right hand side of the equation satisfied by $f^{P_{l'}} - \tilde{f}_j^{P_{l'}}$, it holds that

$$\lim_{j \to +\infty} (f^{P_{l'}} - \tilde{f}_j^{P_{l'}}) = 0,$$

uniformly with respect to l' and t. One then proves by finite induction that for J given,

$$\lim_{l' \to +\infty, l'' \to +\infty} (\tilde{f}_J^{P_{l'}} - \tilde{f}_J^{P_{l''}}) = 0$$

in L^1-sense.

Also $f = \lim f^{P_l} \in L^1$, and

$$\frac{\partial f}{\partial t} + f(n_c \int B\chi\delta(p - p_2 - p_3)\delta(E_p - E_2 - E_3)dp_2 dp_3 + h) = \tilde{C}_g(f, n_c) + 2n_c g, \quad f(0, p) = f_i(p).$$

But the L^1-convergence of (f^{P_l}) implies that

$$g = \lim_{l\to\infty} g^{P_l} = \lim_{l\to\infty} \int B\chi\chi_{|p|<P}\delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p)f_1^P dp_{12}$$
$$= \int B\chi\delta(p_1 - p_2 - p)\delta(E_1 - E_2 - E_p)f_1 dp_{12},$$
$$h = \lim_{l\to\infty} \tilde{C}_l(f^{P_l}, n_c^{P_l}) = \tilde{C}_l(f, n_c).$$

And so (f, n_c) satisfies the system of equations on $[0, T_0]$ in mild form with the $|p|^{2+\gamma}$-moment of f bounded in L^1.
This can be continued up to any time T, since $\inf_{t\in[0,T[} n_c(t) > 0$ for any time interval $[0, T[$ where (f, n_c) exists. The C^1-properties of f, n_c with respect to time, follow as above for f^P, n_c^P.