Fluid Approximations from the Boltzmann Equation for Domains with Boundary

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Introduction

We study some fluid approximations derived from the Boltzmann equation over a smooth bounded spatial domain $\Omega \subset \mathbb{R}^D$. Our focus will be on boundary conditions.

1. We establish the acoustic limit starting from DiPerna-Lions solutions. (Jiang-L-Masmoudi, 2010)

2. We present linearized Navier-Stokes approximations derived formally from the linearized Boltzmann equation.

Acoustic System

After a suitable choice of units and Galilean frame, the acoustic system governs the fluctuations in mass density $\rho(x,t)$, bulk velocity $u(x,t)$, and temperature $\theta(x,t)$ over $\Omega \times \mathbb{R}_+$ by the initial-value problem

$$\begin{aligned} \partial_t \rho + \nabla_x \cdot u = 0\,, &\qquad \rho(x,0) = \rho^{\text{in}}(x)\,, \\ \partial_t u + \nabla_x (\rho + \theta) = 0\,, &\qquad u(x,0) = u^{\text{in}}(x)\,, \\ \tfrac{D}{2}\partial_t \theta + \nabla_x \cdot u = 0\,, &\qquad \theta(x,0) = \theta^{\text{in}}(x)\,, \end{aligned} \tag{1}$$

subject to the impermeable boundary condition

$$u \cdot \mathrm{n} = 0\,, \quad \text{on } \partial\Omega\,, \tag{2}$$

where $\mathrm{n}(x)$ is the unit outward normal at $x \in \partial\Omega$. This is one of the simplest fluid dynamical systems, being essentially the wave equation.

The acoustic system can be derived from the Boltzmann equation for densities $F(v,x,t)$ over $\mathbb{R}^D \times \Omega \times \mathbb{R}_+$ that are near the global Maxwellian

$$M(v) = (2\pi)^{-\frac{D}{2}} \exp\Big(-\tfrac{1}{2}|v|^2\Big)\,. \tag{3}$$

We consider families of densities in the form $F_\epsilon(v,x,t) = M(v)G_\epsilon(v,x,t)$ where the $G_\epsilon(v,x,t)$ are governed over $\mathbb{R}^D \times \Omega \times \mathbb{R}_+$ by the scaled Boltzmann initial-value problem

$$\partial_t G_\epsilon + v\cdot\nabla_x G_\epsilon = \frac{1}{\epsilon}\mathcal{Q}(G_\epsilon, G_\epsilon)\,, \qquad G_\epsilon(v,x,0) = G_\epsilon^{\text{in}}(v,x)\,. \tag{4}$$

Here ϵ is the Knudsen number while $\mathcal{Q}(G_\epsilon, G_\epsilon)$ is given by

$$\mathcal{Q}(G_\epsilon, G_\epsilon) = \iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} \Big(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\Big)\, b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{5}$$

where $b(\omega, v_1 - v) > 0$ a.e. while $G_{\epsilon 1}$, G'_ϵ, and $G'_{\epsilon 1}$ denote $G_\epsilon(\,\cdot\,, x, t)$ evaluated at v_1, $v' = v + \omega\omega\cdot(v_1 - v)$, and $v'_1 = v - \omega\omega\cdot(v_1 - v)$ respectively.

We impose a Maxwell reflection boundary condition on $\partial\Omega$ of the form

$$\mathbf{1}_{\Sigma_+} G_\epsilon \circ \mathsf{R} = (1-\alpha)\,\mathbf{1}_{\Sigma_+} G_\epsilon + \alpha\,\mathbf{1}_{\Sigma_+} \sqrt{2\pi}\left\langle \mathbf{1}_{\Sigma_+} v \cdot n\, G_\epsilon \right\rangle . \tag{6}$$

Here $(G_\epsilon \circ \mathsf{R})(v,x,t) = G_\epsilon(\mathsf{R}(x)v,x,t)$ where $\mathsf{R}(x) = I - 2\mathsf{n}(x)\mathsf{n}(x)^T$ is the specular reflection matrix, $\alpha \in [0,1]$ is the accommodation coefficient, $\mathbf{1}_{\Sigma_+}$ is the indicator function of the so-called outgoing boundary set

$$\Sigma_+ = \left\{ (v,x) \in \mathbb{R}^D \times \partial\Omega \,:\, v \cdot \mathsf{n}(x) > 0 \right\} , \tag{7}$$

and $\langle \cdot \rangle$ denotes the average

$$\langle \xi \rangle = \int_{\mathbb{R}^D} \xi(v)\, M(v)\, \mathrm{d}v \, . \tag{8}$$

Because $\sqrt{2\pi}\left\langle \mathbf{1}_{\Sigma_+} v \cdot \mathsf{n} \right\rangle = 1$, it seen from (6) that on $\partial\Omega$ the flux is

$$\begin{aligned} \langle v \cdot \mathsf{n}\, G_\epsilon \rangle &= \left\langle \mathbf{1}_{\Sigma_+} v \cdot \mathsf{n} \left(G_\epsilon - G_\epsilon \circ \mathsf{R} \right) \right\rangle \\ &= \alpha \left\langle \mathbf{1}_{\Sigma_+} v \cdot \mathsf{n} \left(G_\epsilon - \sqrt{2\pi} \left\langle \mathbf{1}_{\Sigma_+} v \cdot \mathsf{n}\, G_\epsilon \right\rangle \right) \right\rangle = 0 \, . \end{aligned} \tag{9}$$

Formal Derivation

Fluid regimes are those in which ϵ is small. The acoustic system can be derived formally from the scaled Boltzmann equation for families $G_\epsilon(v,x,t)$ that are scaled so that

$$G_\epsilon = 1 + \delta_\epsilon g_\epsilon \,, \qquad G_\epsilon^{\text{in}} = 1 + \delta_\epsilon g_\epsilon^{\text{in}} \,, \tag{10}$$

where

$$\delta_\epsilon \to 0 \qquad \text{as} \qquad \epsilon \to 0 \,, \tag{11}$$

and the fluctuations g_ϵ and g_ϵ^{in} converge in the sense of distributions to $g \in L^\infty(\mathrm{d}t; L^2(M\,\mathrm{d}v\,\mathrm{d}x))$ and $g^{\text{in}} \in L^2(M\,\mathrm{d}v\,\mathrm{d}x)$ respectively as $\epsilon \to 0$.

One finds that g has the infinitesimal Maxwellian form

$$g = \rho + v \cdot u + \left(\tfrac{1}{2}|v|^2 - \tfrac{D}{2}\right)\theta \,, \tag{12}$$

where $(\rho, u, \theta) \in L^\infty(\mathrm{d}t; L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ solve the acoustic system (1) and boundary condition (2) with initial data given by

$$\rho^{\mathrm{in}} = \langle g^{\mathrm{in}} \rangle \,, \qquad u^{\mathrm{in}} = \langle v \, g^{\mathrm{in}} \rangle \,, \qquad \theta^{\mathrm{in}} = \left\langle \left(\tfrac{1}{D}|v|^2 - 1\right) g^{\mathrm{in}} \right\rangle . \tag{13}$$

The boundary condition (2) is obtained by passing to the limit in the boundary flux relation (9) to see

$$0 = \langle v \cdot \mathrm{n} \, g_\epsilon \rangle \to \langle v \cdot \mathrm{n} \, g \rangle \,,$$

We thereby find that $\langle v \cdot \mathrm{n} \, g \rangle = 0$, and finally by using the infinitesimal Maxwellian form (12) get the impermeable boundary condition (2),

$$u \cdot \mathrm{n} = 0 \,.$$

The program initiated with Claude Bardos and Francois Golse in 1989 seeks to justify fluid dynamical limits for Boltzmann equations in the setting of DiPerna-Lions renormalized solutions, which are the only temporally global, large data solutions available.

The main obstruction to carrying out this program is that DiPerna-Lions solutions are not known to satisfy many properties that one formally expects for solutions of the Boltzmann equation. For example, they are not known to satisfy the formally expected local conservations laws of momentum and energy. Moreover, their regularity is poor. The justification of fluid dynamical limits in this setting is therefore not easy.

The acoustic limit was first established in this setting by Bardos-Golse-L (2000) over a periodic domain. There idea introduced there was to pass to the limit in approximate local conservations laws which are satified by DiPerna-Lions solutions. One then shows that the so-called conservation defects vanish as the Knudsen number ϵ vanishes, thereby establishing the local conservation laws in the limit. This was done using only relative entropy estimates, which restricted the result to collision kernels that are bounded and to fluctuations scaled so that

$$\delta_\epsilon \longrightarrow 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon}|\log(\delta_\epsilon)| \longrightarrow 0 \quad \text{as} \quad \epsilon \longrightarrow 0\,, \tag{14}$$

which is far from the formally expected optimal scaling (11), $\delta_\epsilon \longrightarrow 0$.

In Golse-L (2002) the local conservation defects were removed using new dissipation rate estimates. This allowed the treatment of collision kernels that for some $C_b < \infty$ and $\beta \in [0,1)$ satisfied

$$\int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\,\mathrm{d}\omega \leq C_b\Big(1 + |v_1 - v|^2\Big)^{\beta}, \tag{15}$$

and of fluctuations scaled so that

$$\delta_\epsilon \to 0 \quad \text{and} \quad \frac{\delta_\epsilon}{\epsilon^{1/2}}|\log(\delta_\epsilon)|^{\beta/2} \to 0 \quad \text{as} \quad \epsilon \to 0\,. \tag{16}$$

The above class of collision kernels includes all classical kernels that are derived from Maxwell or hard potentials and that satisfy a weak small deflection cutoff. The scaling given by (16) is much less restrictive than that given by (14), but is far from the formally optimal scaling (11). Finally, only periodic domains are treated.

Here we improve the result of Golse-L (2002) in three ways. First, we apply estimates from L-Masmoudi (2010) to treat a broader class of collision kernels that includes those derived from soft potentials. Second, we improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Finally, we treat domains with a boundary and use new estimates to derive the boundary condition (2) in the limit.

The L^1 velocity averaging theory of Golse and Saint-Raymond (2002) is used through the nonlinear compactness estimate of L-Masmoudi (2010) to improve the scaling of the fluctuations to $\delta_\epsilon = O(\epsilon^{1/2})$. Without it we would only be able to improve the scaling to $\delta_\epsilon = o(\epsilon^{1/2})$. This is the first time the L^1 averaging theory has played any role in an acoustic limit theorem, albeit for a modest improvement. We remark that velocity averaging theory plays no role in establishing the Stokes limit with its formally expected optimal scaling of $\delta_\epsilon = o(\epsilon)$.

We treat domains with boundary in the setting of Mischler (2002/2010), who extended DiPerna-Lions theory to bounded domains with a Maxwell reflection boundary condition. He showed that these boundary conditions are satisfied in a *renormalized* sense. This means we cannot deduce that $\langle v \cdot \mathrm{n}\, g_\epsilon \rangle \to 0$ as $\epsilon \to 0$ to derive the boundary condition (2), as we did formally.

Masmoudi and Saint-Raymond (2003) derived boundary conditions in the Stokes limit. However neither these estimates nor their recent extension to the Navier-Stokes limit by Jiang-Masmoudi can handle the acoustic limit. Rather, we develop new boundary *a priori* estimates to obtain a weak form of the boundary condition (2) in this limit. In doing so, we treat a broader class of collision kernels than was treated earlier.

We remark that establishing the acoustic limit with its formally expected optimal scaling of the fluctuation size, $\delta_\epsilon \to 0$, is still open. This gap must be bridged before one can hope to fully establish the compressible Euler limit starting from DiPerna-Lions solutions to the Boltzmann equation.

In contrast, optimal scaling can be obtained within the framework of classical solutions by using the nonlinear energy method developed by Guo. This has been done recently by Guo-Jang-Jiang (2009).

Framework

Let $\Omega \subset \mathbb{R}^D$ be a bounded domain with smooth boundary $\partial\Omega$. Let $\mathsf{n}(x)$ denote the outward unit normal vector at $x \in \partial\Omega$ and $\mathrm{d}\sigma_x$ denote the Lebesgue measure on $\partial\Omega$. The phase space domain associated with Ω is $\mathcal{O} = \mathbb{R}^D \times \Omega$, which has boundary $\partial\mathcal{O} = \mathbb{R}^D \times \partial\Omega$. Let Σ_+ and Σ_- denote the outgoing and incoming subsets of $\partial\mathcal{O}$ defined by

$$\Sigma_\pm = \{(v, x) \in \partial\mathcal{O} \,:\, \pm v \cdot \mathsf{n}(x) > 0\} \,.$$

The global Maxwellian $M(v)$ given by (3) corresponds to the spatially homogeneous fluid state with density and temperature equal to 1 and bulk velocity equal to 0. The boundary condition (6) corresponds to a wall temperature of 1, so that $M(v)$ is the unique equilibrium of the fluid. Associated with the initial data G_ϵ^{in} we have the normalization

$$\int_\Omega \langle G_\epsilon^{\mathrm{in}} \rangle \, \mathrm{d}x = 1 \,. \tag{17}$$

Assumptions on the Collision Kernel

The kernel $b(\omega, v_1 - v)$ associated with the collision operator (5) is positive almost everywhere.

The Galilean invariance of the collisional physics implies that b has the classical form

$$b(\omega, v_1 - v) = |v_1 - v| \, \Sigma(|\omega \cdot n|, |v_1 - v|) \,, \tag{18}$$

where $n = (v_1 - v)/|v_1 - v|$ and Σ is the specific differential cross-section.

We make five additional technical assumptions regarding b that are adopted from L-Masmoudi (2010).

Our *first technical assumption* is that the collision kernel b satisfies the requirements of the DiPerna-Lions theory. That theory requires that b be locally integrable with respect to $\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$, and that it satisfies

$$\lim_{|v|\to\infty} \frac{1}{1+|v|^2} \int_K \bar{b}(v_1 - v)\, \mathrm{d}v_1 = 0 \quad \text{for every compact } K \subset \mathbb{R}^D\,, \tag{19}$$

where $\bar{b}$ is defined by

$$\bar{b}(v_1 - v) \equiv \int_{\mathbb{S}^{D-1}} b(\omega, v_1 - v)\, \mathrm{d}\omega\,. \tag{20}$$

Galilean symmetry (18) implies that $\bar{b}$ is a function of $|v_1 - v|$ only.

Our *second technical assumption* regarding b is that the attenuation coefficient a, which is defined by

$$a(v) \equiv \int_{\mathbb{R}^D} \overline{b}(v_1 - v)\, M_1 \mathrm{d}v_1\,, \tag{21}$$

is bounded below as

$$C_a \Big(1 + |v|^2\Big)^{\beta_a} \leq a(v) \quad \text{for some } C_a > 0 \text{ and } \beta_a \in \mathbb{R}\,. \tag{22}$$

Galilean symmetry (18) implies that a is a function of $|v|$ only.

Our *third technical assumption* regarding b is that there exists $s \in (1, \infty]$ and $C_b \in (0, \infty)$ such that

$$\left(\int_{\mathbb{R}^D} \left| \frac{\overline{b}(v_1 - v)}{a(v_1)\, a(v)} \right|^s a(v_1)\, M_1 \mathrm{d}v_1 \right)^{\frac{1}{s}} \leq C_b\,. \tag{23}$$

Because this bound is uniform in v, we may take C_b to be the supremum over v of the left-hand side of (23).

Our *fourth technical assumption* regarding b is that the operator

$$\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v) \quad \text{is compact}\,, \tag{24}$$

where

$$\mathcal{K}^+\tilde{g} = \frac{1}{2a}\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D}\Big(\tilde{g}' + \tilde{g}_1'\Big)\, b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1\mathrm{d}v_1\,.$$

We remark that $\mathcal{K}^+ : L^2(aM\mathrm{d}v) \to L^2(aM\mathrm{d}v)$ is always bounded with $\|\mathcal{K}^+\| \leq 1$.

Our *fifth technical assumption* regarding b is that for every $\delta > 0$ there exists C_δ such that $\overline{b}$ satisfies

$$\frac{\overline{b}(v_1 - v)}{1 + \delta\,\dfrac{\overline{b}(v_1 - v)}{1 + |v_1 - v|^2}} \leq C_\delta\Big(1 + a(v_1)\Big)\Big(1 + a(v)\Big) \quad \text{for every } v_1, v \in \mathbb{R}^D\,. \tag{25}$$

The above assumptions are satisfied by all the classical collision kernels with a weak small deflection cutoff that derive from a repulsive intermolecular potential of the form c/r^k with $k > 2\frac{D-1}{D+1}$. This includes all the classical collision kernels to which the DiPerna-Lions theory applies. Kernels that satisfy (15) clearly satisfy (19). If they moreover satisfy (22) with $\beta_a = \beta$ then they also satisfy (23) and (25).

Because the kernel b satisfies (19), it can be normalized so that

$$\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1\,\mathrm{d}v_1\, M\,\mathrm{d}v = 1\,.$$

Because $\mathrm{d}\mu = b(\omega, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\, M \mathrm{d}v$ is a positive unit measure on $\mathbb{S}^{D-1} \times \mathbb{R}^D \times \mathbb{R}^D$, we denote by $\langle\!\langle \Xi \rangle\!\rangle$ the average over this measure of any integrable function $\Xi = \Xi(\omega, v_1, v)$

$$\langle\!\langle \Xi \rangle\!\rangle = \iiint_{\mathbb{S}^{D-1}\times\mathbb{R}^D\times\mathbb{R}^D} \Xi(\omega, v_1, v)\,\mathrm{d}\mu\,. \tag{26}$$

DiPerna-Lions-Mischler Theory

We will work in the framework of DiPerna-Lions solutions to the scaled Boltzmann equation on the phase space $\mathcal{O} = \mathbb{R}^D \times \Omega$

$$\begin{aligned} \partial_t G_\epsilon + v \cdot \nabla_x G_\epsilon &= \frac{1}{\epsilon} \mathcal{Q}(G_\epsilon, G_\epsilon) \quad && \text{on} \quad \mathcal{O} \times \mathbb{R}_+ \,, \\ G_\epsilon(v, x, 0) &= G_\epsilon^{\text{in}}(v, x) && \text{on} \quad \mathcal{O} \,, \end{aligned} \tag{27}$$

with the Maxwell reflection boundary condition (6) which can be expressed as

$$\gamma_- G_\epsilon = (1 - \alpha) L(\gamma_+ G_\epsilon) + \alpha \langle \gamma_+ G_\epsilon \rangle_{\partial\Omega} \quad \text{on} \quad \Sigma_- \times \mathbb{R}_+ \,, \tag{28}$$

where $\gamma_\pm G_\epsilon$ denote the traces of G_ϵ on the outgoing and incoming sets $\Sigma_\pm$.

Here the local reflection operator L is defined to act on any $|v \cdot \mathsf{n}| M \, \mathrm{d}v \, \mathrm{d}\sigma$-measurable function ϕ over $\partial\mathcal{O}$ by

$$L\phi(v, x) = \phi(\mathsf{R}(x)v, x) \quad \text{for almost every } (v, x) \in \partial\mathcal{O} \,,$$

where $\mathsf{R}(x)v = v - 2v \cdot \mathsf{n}(x)\mathsf{n}(x)$ is the specular reflection of v, while the diffuse reflection operator is defined as

$$\langle \phi \rangle_{\partial\Omega} = \sqrt{2\pi} \int_{v \cdot \mathsf{n}(x) > 0} \phi(v, x) \, v \cdot \mathsf{n}(x) \, M \mathrm{d}v \,.$$

DiPerna-Lions theory requires that both the equation and boundary conditions in (27) should be understood in the renormalized sense, see (44) and (48). These solutions were initially constructed by DiPerna and Lions over the whole space $\mathbb{R}^D$ for any initial data satisfying natural physical bounds. For bounded domain case, Mischler recently developed a theory to treat the Maxwell reflection boundary condition.

The DiPerna-Lions theory does not yield solutions that are known to solve the Boltzmann equation in the usual sense of weak solutions. Rather, it gives the existence of a global weak solution to a class of formally equivalent initial value problems that are obtained by multiplying (27) by $\Gamma'(G_\epsilon)$, where Γ' is the derivative of an admissible function Γ:

$$(\partial_t + v \cdot \nabla_x)\Gamma(G_\epsilon) = \frac{1}{\epsilon}\Gamma'(G_\epsilon)\mathcal{Q}(G_\epsilon, G_\epsilon) \quad \text{on} \quad \mathcal{O} \times \mathbb{R}_+ . \tag{29}$$

Here a function $\Gamma : [0,\infty) \to \mathbb{R}$ is called admissible if it is continuously differentiable and for some $C_\Gamma < \infty$ its derivative satisfies

$$|\Gamma'(Z)| \le \frac{C_\Gamma}{\sqrt{1+Z}} \quad \text{for every } Z \in [0,\infty) .$$

The solutions are nonnegative and lie in $C([0,\infty); \text{w-}L^1(M \, \mathrm{d}v \, \mathrm{d}x))$, where the prefix “w-” on a space indicates that the space is endowed with its weak topology.

Mischler (2010) extended DiPerna-Lions theory to domains with a boundary on which the Maxwell reflection boundary condition (28) is imposed. This required the proof of a so-called trace theorem that shows that the restriction of G_ϵ to $\partial\mathcal{O} \times \mathbb{R}_+$, denoted γG_ϵ, makes sense. In particular, Mischler showed that γG_ϵ lies in the set of all $|v \cdot \mathrm{n}| M \,\mathrm{d}v \,\mathrm{d}\sigma \,\mathrm{d}t$-measurable functions over $\partial\mathcal{O} \times \mathbb{R}_+$ that are finite almost everywhere, which we denote $L^0(|v \cdot \mathrm{n}| M \,\mathrm{d}v \,\mathrm{d}\sigma \,\mathrm{d}t)$. He then defines $\gamma_\pm G_\epsilon = \mathbf{1}_{\Sigma_\pm} \gamma G_\epsilon$. He proves the following.

Theorem. *(DiPerna-Lions-Mischler Renormalized Solutions)* Let b be a collision kernel that satisfies the assumptions given earlier. Fix $\epsilon > 0$. Let G_ϵ^{in} be any initial data in the entropy class

$$E(M\,\mathrm{d}v\,\mathrm{d}x) = \left\{G_\epsilon^{\mathrm{in}} \geq 0 \;:\; H(G_\epsilon^{\mathrm{in}}) < \infty\right\}, \tag{30}$$

where the relative entropy functional is given by

$$H(G) = \int_\Omega \langle \eta(G) \rangle \,\mathrm{d}x \quad \text{with} \quad \eta(G) = G\log(G) - G + 1\,.$$

Then there exists a $G_\epsilon \geq 0$ in $C([0,\infty); \mathrm{w}\text{-}L^1(M\,\mathrm{d}v\,\mathrm{d}x))$ with $\gamma G_\epsilon \geq 0$ in $L^0(|v\cdot \mathrm{n}|M\,\mathrm{d}v\,\mathrm{d}\sigma\,\mathrm{d}t)$ such that:

- G_ϵ satisfies the global entropy inequality

$$H(G_\epsilon(t)) + \int_0^t \left[\frac{1}{\epsilon} R(G_\epsilon(s)) + \frac{\alpha}{\sqrt{2\pi}} \mathcal{E}(\gamma_+ G_\epsilon(s))\right] \mathrm{d}s \le H(G_\epsilon^{\mathrm{in}}) \quad \text{for every } t > 0\,, \tag{31}$$

where the entropy dissipation rate functional is given by

$$R(G) = \frac{1}{4} \int_\Omega \left\langle\!\!\left\langle \log\left(\frac{G_1' G'}{G_1 G}\right) \left(G_1' G' - G_1 G\right) \right\rangle\!\!\right\rangle \mathrm{d}x\,, \tag{32}$$

and the so-called Darrozès-Guiraud information is given by

$$\mathcal{E}(\gamma_+ G) = \int_{\partial\Omega} \left[\left\langle \eta(\gamma_+ G) \right\rangle_{\partial\Omega} - \eta\left(\langle \gamma_+ G \rangle_{\partial\Omega}\right)\right] \mathrm{d}\sigma\,; \tag{33}$$

- G_ϵ satisfies

$$\begin{aligned}&\int_\Omega \langle \Gamma(G_\epsilon(t_2))\, Y\rangle \,\mathrm{d}x - \int_\Omega \langle \Gamma(G_\epsilon(t_1))\, Y\rangle \,\mathrm{d}x \\ &+ \int_{t_1}^{t_2}\!\!\int_{\partial\Omega} \langle \Gamma(\gamma G_\epsilon)\, Y\,(v\cdot \mathrm{n})\rangle \,\mathrm{d}\sigma\,\mathrm{d}t - \int_{t_1}^{t_2}\!\!\int_\Omega \langle \Gamma(G_\epsilon)\, v\cdot\nabla_x Y\rangle \,\mathrm{d}x\,\mathrm{d}t \\ &= \frac{1}{\epsilon}\int_{t_1}^{t_2}\!\!\int_\Omega \langle \Gamma'(G_\epsilon)\,\mathcal{Q}(G_\epsilon, G_\epsilon)\, Y\rangle \,\mathrm{d}x\,\mathrm{d}t\,,\end{aligned} \tag{34}$$

  for every admissible function Γ, every $Y \in C^1 \cap L^\infty(\mathbb{R}^D \times \bar{\Omega})$, and every $[t_1, t_2] \subset [0, \infty]$;

- G_ϵ satisfies

$$\begin{gathered}\gamma_- G_\epsilon = (1-\alpha) L(\gamma_+ G_\epsilon) + \alpha \langle \gamma_+ G_\epsilon\rangle_{\partial\Omega} \\ \text{almost everywhere on} \quad \Sigma_- \times \mathbb{R}_+\,.\end{gathered} \tag{35}$$

Remark. Because the γG_ϵ is only known to exist in $L^0(|v \cdot \mathrm{n}| M \,\mathrm{d}v \,\mathrm{d}\sigma \,\mathrm{d}t)$ rather than in $L^1_{loc}(\mathrm{d}t; L^1(|v \cdot \mathrm{n}| M \,\mathrm{d}v \,\mathrm{d}\sigma))$, we cannot conclude from the boundary condition (35) that

$$\langle v \, \gamma G_\epsilon \rangle \cdot \mathrm{n} = 0 \quad \text{on} \quad \partial\Omega \, . \tag{36}$$

Indeed, we cannot even conclude that the boundary mass-flux $\langle v \, \gamma G_\epsilon \rangle \cdot \mathrm{n}$ is defined on $\partial\Omega$. Moreover, in contrast to DiPerna-Lions theory over the whole space or periodic domains, it is not asserted that G_ϵ satisfies the weak form of the local mass conservation law

$$\int_\Omega \chi \, \langle G_\epsilon(t_2) \rangle \,\mathrm{d}x - \int_\Omega \chi \, \langle G_\epsilon(t_1) \rangle \,\mathrm{d}x - \int_{t_1}^{t_2} \int_\Omega \nabla_x \chi \cdot \langle v \, G_\epsilon \rangle \,\mathrm{d}x \,\mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega}) \, . \tag{37}$$

If this were the case, it would allow a great simplification the proof of our main result. Rather, we will employ the boundary condition (35) inside an approximation to (37) that has a well-defined boundary flux.

Main Result

We will consider families G_ϵ of DiPerna-Lions renormalized solutions to (27) such that $G_\epsilon^{\text{in}} \geq 0$ satisfies the entropy bound

$$H(G_\epsilon^{\text{in}}) \leq C^{\text{in}} \delta_\epsilon^2 \tag{38}$$

for some $C^{\text{in}} < \infty$ and $\delta_\epsilon > 0$ that satisfies the scaling $\delta_\epsilon \to 0$ as $\epsilon \to 0$.

The value of $H(G)$ provides a natural measure of the proximity of G to the equilibrium $G = 1$. We define the families g_ϵ^{in} and g_ϵ of fluctuations about $G = 1$ by the relations

$$G_\epsilon^{\text{in}} = 1 + \delta_\epsilon g_\epsilon^{\text{in}} \,, \qquad G_\epsilon = 1 + \delta_\epsilon g_\epsilon \,. \tag{39}$$

One easily sees that H asymptotically behaves like half the square of the L^2-norm of these fluctuations as $\epsilon \to 0$.

Hence, the entropy bound (38) combined with the entropy inequality (31) is consistent with these fluctuations being of order 1. Just as the relative entropy H controls the fluctuations g_ϵ, the dissipation rate R given by (32) controls the scaled collision integrals defined by

$$q_\epsilon = \frac{1}{\sqrt{\epsilon}\delta_\epsilon}\Big(G'_{\epsilon 1}G'_\epsilon - G_{\epsilon 1}G_\epsilon\Big)\,.$$

Here we only state the weak acoustic limit theorem because the corresponding strong limit theorem is analogous to that stated in Golse-L (2002) and its proof based on the weak limit theorem and relative entropy convergence is essentially the same.

Theorem. *(Weak Acoustic Limit)* Let b be a collision kernel that satisfies the assumptions given earlier.

Let G_ϵ^{in} be a family in the entropy class $E(M\,\mathrm{d}v\,\mathrm{d}x)$ that satisfies the normalization (17) and the entropy bound (38) for some $C^{\text{in}} < \infty$ and $\delta_\epsilon > 0$ satisfies the scaling

$$\delta_\epsilon = O\Big(\sqrt{\epsilon}\Big)\,.$$

Assume, moreover, that for some $(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) \in L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R})$ the family of fluctuations g_ϵ^{in} defined by (39) satisfies

$$(\rho^{\text{in}}, u^{\text{in}}, \theta^{\text{in}}) = \lim_{\epsilon\to 0} \Big(\langle g_\epsilon^{\text{in}}\rangle\,,\,\langle v\,g_\epsilon^{\text{in}}\rangle\,,\,\Big\langle\Big(\tfrac{1}{D}|v|^2 - 1\Big)\,g_\epsilon^{\text{in}}\Big\rangle\Big) \quad \text{in the sense of distributions}\,. \tag{40}$$

Let G_ϵ be any family of DiPerna-Lions-Mischler renormalized solutions to the Boltzmann equation (27) that have G_ϵ^{in} as initial values.

Then, as $\epsilon \to 0$, the family of fluctuations g_ϵ defined by (39) satisfies

$$g_\epsilon \to \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta \quad \text{in w-}L^1_{loc}(\mathrm{d}t; \text{w-}L^1((1+|v|^2)M\,\mathrm{d}v\,\mathrm{d}x))\,, \tag{41}$$

where $(\rho, u, \theta) \in C([0,\infty); L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ is the unique solution to the acoustic system (1) that satisfies the impermeable boundary condition (2) and has initial data $(\rho^{\mathrm{in}}, u^{\mathrm{in}}, \theta^{\mathrm{in}})$ obtain from (40). In addition, ρ satisfies

$$\int_\Omega \rho\,\mathrm{d}x = 0\,. \tag{42}$$

This result improves upon the acoustic limit result in three ways:

1. Its assumption on the collision kernel b is the same as L-Masmoudi, so it treats a broader class of cut-off kernels than in Golse-L (2002). In particular, it treats kernals derived from soft potentials.

2. Its scaling assumption is $\delta_\epsilon = O(\sqrt{\epsilon})$, which is certainly better than the scaling assumption (16) used in Golse-L. This is still a long way from that required by the formal derivation.

3. We derive a weak form of the boundary condition $u \cdot \mathrm{n} = 0$. It is the first time such a boundary condition is derived for the acoustic system.

Proof of Main Theorem

In order to derive the fluid equations with boundary conditions, we need to pass to the limit in approximate local conservation laws built from the renormalized Boltzmann equation (29). We choose the renormalization

$$\Gamma(Z) = \frac{Z-1}{1+(Z-1)^2}\,. \tag{43}$$

After dividing by δ_ϵ, equation (29) becomes

$$\partial_t \tilde{g}_\epsilon + v\cdot\nabla_x \tilde{g}_\epsilon = \frac{1}{\sqrt{\epsilon}}\,\Gamma'(G_\epsilon)\iint_{\mathbb{S}^{D-1}\times\mathbb{R}^D} q_\epsilon\, b(\omega\,, v_1 - v)\,\mathrm{d}\omega\, M_1 \mathrm{d}v_1\,, \tag{44}$$

where $\tilde{g}_\epsilon = \Gamma(G_\epsilon)/\delta_\epsilon$. By introducing $N_\epsilon = 1 + \delta_\epsilon^2 g_\epsilon^2$, we can write

$$\tilde{g}_\epsilon = \frac{g_\epsilon}{N_\epsilon}\,, \qquad \Gamma'(G_\epsilon) = \frac{2}{N_\epsilon^2} - \frac{1}{N_\epsilon}\,. \tag{45}$$

When moment of the renormalized Boltzmann equation (44) is formally taken with respect to any $\zeta \in \mathrm{span}\{1\,, v_1\,, \cdots\,, v_D\,, |v|^2\}$, one obtains

$$\partial_t \langle \zeta\, \tilde{g}_\epsilon \rangle + \nabla_x \cdot \langle v\, \zeta\, \tilde{g}_\epsilon \rangle = \frac{1}{\sqrt{\epsilon}} \left\langle\!\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle . \tag{46}$$

Every DiPerna-Lions solution satisfies (46) in the sense that for every $\chi \in C^1(\Omega)$ and every $[t_1, t_2] \subset [0, \infty)$ it satisfies

$$\begin{aligned} &\int_\Omega \chi\, \langle \zeta\, \tilde{g}_\epsilon(t_2) \rangle\, \mathrm{d}x - \int_\Omega \chi\, \langle \zeta\, \tilde{g}_\epsilon(t_1) \rangle\, \mathrm{d}x + \int_{t_1}^{t_2}\!\!\int_{\partial\Omega} \chi\, \langle v\, \zeta\, \gamma \tilde{g}_\epsilon \rangle \cdot \mathrm{n}\, \mathrm{d}\sigma\, \mathrm{d}t \\ &- \int_{t_1}^{t_2}\!\!\int_\Omega \nabla_x \chi \cdot \langle v\, \zeta\, \tilde{g}_\epsilon \rangle\, \mathrm{d}x\, \mathrm{d}t = \int_{t_1}^{t_2}\!\!\int_\Omega \chi\, \frac{1}{\sqrt{\epsilon}} \left\langle\!\!\left\langle \zeta\, \Gamma'(G_\epsilon)\, q_\epsilon \right\rangle\!\!\right\rangle \mathrm{d}x\, \mathrm{d}t\, . \end{aligned} \tag{47}$$

Moreover, from (35) the boundary condition is understood in the renormalized sense:

$$\gamma_-\tilde{g}_\epsilon = \frac{(1-\alpha)L\gamma_+ g_\epsilon + \alpha\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}}{1+\delta_\epsilon^2[(1-\alpha)L\gamma_+ g_\epsilon + \alpha\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}]^2} \quad \text{on } \Sigma_- \times \mathbb{R}_+\,, \tag{48}$$

where the equality holds almost everywhere. We will pass to the limit in the weak form (47).

The Main Theorem will be proved in two steps: the interior equations will be established first and the boundary condition second.

Establishing the Acoustic System

The acoustic system (1) is justified in the interior of Ω by showing that the limit of (47) as $\epsilon \longrightarrow 0$ is the weak form of the acoustic system whenever the test function χ vanishes on $\partial\Omega$. We prove that the conservation defect on the right-hand side of (47) vanishes as $\epsilon \longrightarrow 0$. The proof of the analogous result in Golse-L (2002) must be modified in order to include the case $\delta_\epsilon = O(\sqrt{\epsilon})$. The convergence of the density and flux terms is proved essentially the same, so we omit those arguments here. The upshot is that every converging subsequence of the family g_ϵ satisfies

$$g_\epsilon \to \rho + v \cdot u + (\tfrac{1}{2}|v|^2 - \tfrac{D}{2})\theta \quad \text{in w-}L^1_{loc}(\mathrm{d}t; \text{w-}L^1((1 + |v|^2) M \, \mathrm{d}v \, \mathrm{d}x)) \, ,$$

where $(\rho, u, \theta) \in C([0,\infty); \text{w-}L^2(\mathrm{d}x; \mathbb{R} \times \mathbb{R}^D \times \mathbb{R}))$ satisfies for every $[t_1, t_2] \subset [0,\infty)$

$$\int_\Omega \chi\, \rho(t_2)\, \mathrm{d}x - \int_\Omega \chi\, \rho(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\!\!\int_\Omega \nabla_x \chi \cdot u\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall \chi \in C_0^1(\overline{\Omega}), \tag{49a}$$

$$\int_\Omega w \cdot u(t_2)\, \mathrm{d}x - \int_\Omega w \cdot u(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\!\!\int_\Omega \nabla_x \cdot w\, (\rho + \theta)\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall w \in C_0^1(\overline{\Omega}; \mathbb{R}^D), \tag{49b}$$

$$\frac{D}{2} \int_\Omega \chi\, \theta(t_2)\, \mathrm{d}x - \frac{D}{2} \int_\Omega \chi\, \theta(t_1)\, \mathrm{d}x - \int_{t_1}^{t_2}\!\!\int_\Omega \nabla_x \chi \cdot u\, \mathrm{d}x\, \mathrm{d}t = 0 \qquad \forall \chi \in C_0^1(\overline{\Omega}). \tag{49c}$$

This shows that the acoustic system (1) is satisfied in the interior of Ω.

Establishing the Boundary Condition

The more significant step is to justify the impermeable boundary condition (2). Unlike to what is done for the incompressible Stokes and Navier-Stokes limits, here we do not have enough control to pass to the limit in the boundary terms in (47) for the local conservation laws of momentum and energy. We can however do so for the local conservation law of mass — i.e. when $\zeta = 1$. Indeed, we can extend (49a) to

$$\int_\Omega \chi\,\rho(t_2)\,\mathrm{d}x - \int_\Omega \chi\,\rho(t_1)\,\mathrm{d}x - \int_{t_1}^{t_2}\int_\Omega \nabla_x\chi\cdot u\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega})\,. \tag{50}$$

We obtain 42 by setting $\chi = 1$ and $t_1 = 0$ above, and using the fact that the family G_ϵ^{in} satisfies the normalization (17).

Because for every $\chi \in C^1(\overline{\Omega})$ we can find a sequence $\{\chi_n\} \subset C^1_0(\overline{\Omega})$ such that $\chi_n \to \chi$ in $L^2(\mathrm{d}x)$, it follows from (49a) and (49c) that

$$\begin{aligned}
\frac{D}{2}\int_\Omega \chi\,\theta(t_2)\,\mathrm{d}x - \frac{D}{2}\int_\Omega \chi\,\theta(t_1)\,\mathrm{d}x \\
&= \lim_{n\to\infty}\frac{D}{2}\int_\Omega \chi_n\,\theta(t_2)\,\mathrm{d}x - \lim_{n\to\infty}\frac{D}{2}\int_\Omega \chi_n\,\theta(t_1)\,\mathrm{d}x \\
&= \lim_{n\to\infty}\int_\Omega \chi_n\,\rho(t_2)\,\mathrm{d}x - \lim_{n\to\infty}\int_\Omega \chi_n\,\rho(t_1)\,\mathrm{d}x \\
&= \int_\Omega \chi\,\rho(t_2)\,\mathrm{d}x - \int_\Omega \chi\,\rho(t_1)\,\mathrm{d}x\,.
\end{aligned}$$

It thereby follows from (50) that we can extend (49c) to

$$\begin{aligned}
\frac{D}{2}\int_\Omega \chi\,\theta(t_2)\,\mathrm{d}x - \frac{D}{2}\int_\Omega \chi\,\theta(t_1)\,\mathrm{d}x \\
- \int_{t_1}^{t_2}\!\int_\Omega \nabla_x\chi\cdot u\,\mathrm{d}x\,\mathrm{d}t = 0 \qquad \forall \chi \in C^1(\overline{\Omega})\,.
\end{aligned} \tag{51}$$

Finally, because for every $w \in C^1(\overline{\Omega};\mathbb{R}^D)$ such that $w \cdot \mathsf{n} = 0$ on $\partial\Omega$ we can find a sequence $\{w_n\} \subset C^1_0(\overline{\Omega};\mathbb{R}^D)$ such that $w_n \to w$ in $L^2(\mathrm{d}x;\mathbb{R}^D)$ and $\nabla_x \cdot w_n \to \nabla_x \cdot w$ in $L^2(\mathrm{d}x)$, it follows from (49b) that

$$\begin{aligned}\int_\Omega w \cdot u(t_2)\,\mathrm{d}x - \int_\Omega w \cdot u(t_1)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_\Omega w_n \cdot u(t_2)\,\mathrm{d}x - \lim_{n\to\infty} \int_\Omega w_n \cdot u(t_1)\,\mathrm{d}x \\ &= \lim_{n\to\infty} \int_{t_1}^{t_2}\!\int_\Omega \nabla_x \cdot w_n\,(\rho+\theta)\,\mathrm{d}x\,\mathrm{d}t \\ &= \int_{t_1}^{t_2}\!\int_\Omega \nabla_x \cdot w\,(\rho+\theta)\,\mathrm{d}x\,\mathrm{d}t\,.\end{aligned}$$

But this combined with (50) and (51) is the weak formulation of the acoustic system (1) with the boundary condition (2).

This system has a unique solution in $C([0,\infty);\text{w-}L^2(\mathrm{d}x;\mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$, so all converging sequences of the family g_ϵ have this same limit. So this limit must be the strong solution in $C([0,\infty);L^2(\mathrm{d}x;\mathbb{R}\times\mathbb{R}^D\times\mathbb{R}))$. The family of fluctuations g_ϵ therefore converges as asserted by (41).

Remark. The most important difference between the acoustic limit and the incompressible limits (Stokes and Navier-Stokes) is that the compactness of the renormalized traces $\gamma\tilde{g}_\epsilon$ in the acoustic limit case is not available. The pointwise convergence $\delta_\epsilon\tilde{g}_\epsilon \to 0$ a.e. is also unavailable. In contrast, for the incompressible limits the entropy bounds from boundary provide *a priori* estimates on the quantity $\gamma_\epsilon = \gamma_+ g_\epsilon - \mathbf{1}_{\Sigma_+}\langle\gamma_+ g_\epsilon\rangle_{\partial\Omega}$. Specifically, we have the L^2 bound on $\frac{1}{\delta_\epsilon}\frac{\gamma_\epsilon^{(1)}}{n_\epsilon}$ with some renormalizer n_ϵ. However, in the acoustic limit, because of the acoustic scaling, we have only the L^2 bound on $\frac{\gamma_\epsilon^{(1)}}{n_\epsilon}$ which is much weaker than in the incompressible limits cases.

Linearized Boltzmann Setting

Now consider the linearized Boltzmann

$$\partial_t g_\epsilon + v \cdot \nabla_x g_\epsilon + \frac{1}{\epsilon}\mathcal{L} g_\epsilon = 0\,,$$

Here $\mathcal{L}g = -2\mathcal{Q}(1, g)$ as before. This is well-posed in $L^2(M\,\mathrm{d}v\,\mathrm{d}x)$ over a smooth spatial domain Ω with proper boundary conditions. Let

$$\mathbf{m} = \begin{pmatrix} 1 & v_1 & \cdots & v_D & \frac{1}{2}|v|^2 \end{pmatrix}^T .$$

The local conservation laws are then

$$\partial_t \langle \mathbf{m}\, g_\epsilon \rangle + \nabla_x \cdot \langle v\, \mathbf{m}\, g_\epsilon \rangle = 0\,.$$

In the interior of Ω we can approximate g_ϵ by the Enskog expansion

$$\begin{aligned} g_\epsilon = \mathbf{m}^T\boldsymbol{\alpha}_\epsilon - \epsilon\,\mathcal{L}^{-1}\mathbf{m}^T v\cdot\nabla_x\boldsymbol{\alpha}_\epsilon \\ + \epsilon^2\mathcal{L}^{-1}\left(\partial_t + v\cdot\nabla_x\right)\mathcal{L}^{-1}\mathbf{m}^T v\cdot\nabla_x\boldsymbol{\alpha}_\epsilon + O(\epsilon^3)\,, \end{aligned}$$

where $\langle\mathbf{m}\,\mathbf{m}^T\rangle\boldsymbol{\alpha}_\epsilon = \langle\mathbf{m}\,g_\epsilon\rangle$.

If we approximate $\boldsymbol{\alpha}_\epsilon$ by the solution of the acoustic system

$$\begin{aligned} \langle\mathbf{m}\,\mathbf{m}^T\rangle\partial_t\boldsymbol{\alpha}_\epsilon + \langle\mathbf{m}\,\mathbf{m}^T v\rangle\cdot\nabla_x\boldsymbol{\alpha}_\epsilon = 0\,, \\ \langle\mathbf{m}\,\mathbf{m}^T\rangle\boldsymbol{\alpha}_\epsilon\Big|_{t=0} = \langle\mathbf{m}\,g_\epsilon^{\mathrm{in}}\rangle\,, \end{aligned}$$

then we expect to prove that

$$g_\epsilon = \mathbf{m}^T\boldsymbol{\alpha}_\epsilon + O(\epsilon)\,.$$

If we approximate $\boldsymbol{\alpha}_\epsilon$ by the solution of the compressible Stokes system

$$\langle \mathbf{m}\,\mathbf{m}^T\rangle \partial_t \boldsymbol{\alpha}_\epsilon + \langle \mathbf{m}\,\mathbf{m}^T v\rangle \cdot \nabla_x \boldsymbol{\alpha}_\epsilon = \epsilon\, \nabla_x \cdot \left[\langle v\, \mathbf{m}\, \mathcal{L}^{-1} \mathbf{m}^T v\rangle \cdot \nabla_x \boldsymbol{\alpha}_\epsilon \right] ,$$

$$\langle \mathbf{m}\,\mathbf{m}^T\rangle \boldsymbol{\alpha}_\epsilon \Big|_{t=0} = \langle \mathbf{m}\, g_\epsilon^{\mathrm{in}}\rangle - \epsilon\, \langle \mathbf{m}\, v \cdot \nabla_x \mathcal{L}^{-1} g_\epsilon^{\mathrm{in}}\rangle ,$$

then we expect to prove that

$$g_\epsilon = \mathbf{m}^T \boldsymbol{\alpha}_\epsilon - \epsilon\, \mathcal{L}^{-1} \mathbf{m}^T v \cdot \nabla_x \boldsymbol{\alpha}_\epsilon + O(\epsilon^2) .$$

This requires a boundary layer construction through order ϵ.

The “count” for such a construction is correct, unlike for the acoustic approximation. More specifically, the number of conditions needed to insure that the solution of a half-space problem decays is the number of incoming plus the number of tangential characteristic velocities of the acoustic system. This is generally greater than the number of conditions required to make the acoustic system well-posed, but is equal to the number of conditions required to make the compressible Stokes system well-posed.

Open Problems in the BGL Program

1. Acoustic limit with optimal scaling, $\delta_\epsilon \longrightarrow 0$.

2. Compressible Stokes approximation (linearized compressible N-S).

3. Weakly nonlinear/dissipative approximation to compressible N-S.

4. Dominant-balance incompressible approximations (Bardos-L-Ukai-Yang)

5. Bounded domains (Bardos, Jiang, Masmoudi, L, Saint-Raymond, $\cdots$)

Thank You!