Fourier Law and Non-Isothermal Boundary in the Boltzmann Theory

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Steady Boltzmann Equation

Steady Boltzmann Equation

$$v \cdot \nabla_x F = Q(F, F)$$

- $F(x, v)$: density distribution of rarefied gas
- 3 D velocity space $v \in \mathbf{R}^3$
- Ω : bounded, connected domain in $\mathbf{R}^d$ for $d = 1, 2, 3$
- nonlinear Boltzmann operator $Q(F_1, F_2)$:
  - quadratic, bilinear
  - non-local in $v \in \mathbf{R}^3$
  - hard potential $0 \leq \gamma \leq 1$ with angular cut-off
  - collision invariant : $\int_{\mathbf{R}^3} \{1, v, |v|^2\} Q(F, F)(v) dv = 0$
- Knudsen number ~ 1 regime

Non-Isothermal Boundary and Diffusive BC

Wall temperature

$$\theta(x) = \theta_0 + \delta\vartheta(x) \quad \text{on} \quad x \in \partial\Omega$$

Diffusive boundary condition on $x \in \partial\Omega,\ n(x)\cdot v < 0$

$$F(x,v) = \mu^\theta(x,v)\int_{n(x)\cdot u>0} F(x,u)\{n(x)\cdot u\}du$$

Wall Maxwellian

$$\mu^\theta(x,v) = \frac{1}{2\pi\theta(x)^2}\exp\left[-\frac{|v|^2}{2\theta(x)}\right]$$

with $\int_{n(x)\cdot v>0}\mu^\theta(x,v)\{n(x)\cdot v\}dv = 0$

Purpose of This Work

- Analyze the thermal conduction phenomena in the kinetic regime(Knudsen number ~ 1)

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- Analyze the thermal conduction phenomena in the kinetic regime(Knudsen number ~ 1) when the wall temperature do not oscillate too much!

$$\Downarrow$$

- Steady Boltzmann equation with Duffuse BC

$$|\theta(x) - \theta_0| \ll 1, \quad |\vartheta(x)| \leq 1 \text{ and } \delta \ll 1$$

$$\Downarrow$$

$$F_s \sim \mu \ \text{ Regime}$$

Natural Questions and Previous Works

- Existence, Uniqueness, Non-Negativity for Steady Solution
  - S.-H.Yu : existence and stability, Ω is slab (length$\ll 1$), ARMA 2009
  - L.Arkeryd, A.Nouri : Ann. Fac. Sci. Toulouse, Math. 2000 : Existence in L^1-space, Ω is slab
- Regularity (Continuity and Singularity)
  - Y.Guo : for IBVP, Ω convex, continuity away from γ_0 : ARMA 2010
  - C.K : for IBVP, Ω non-convex, singularity formation and propagation : CMP 2011

Natural Questions and Previous Works

- Dynamical Stability
  - L.Desvillettes, C.Villani : polynomial decay in H^k for some BCs : Invent. Math. 2005
  - C.Villani : polynomial decay in H^k, diffusive BC, $\theta \equiv \theta_0$: Mem. AMS 2009
  - Y.Guo : $\theta \equiv \theta_0$, $e^{-\lambda t}-$decay in L^∞ to μ: ARMA 2010
  - S.-H.Yu : $e^{-\lambda t}-$decay in L^∞ to the steady solution : ARMA 2009
- Hydrodynamic Limit
  - R. Esposito, Lebowitz, R.Marra : CMP 1994, J.Stat.Phys. 1995

Theorem : Existence, Uniqueness and Non-Negativity

Let $\Omega \subset \mathbf{R}^d, \; d = 1, 2, 3.$
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For all $M > 0$, there exists $\delta_0 > 0$ such that for $0 < \delta < \delta_0$ in

$$|\theta(x) - \theta_0| \leq \delta, \quad \text{on} \quad x \in \partial\Omega,$$

then there exists a non-negative solution $F_s = M\mu + \sqrt{\mu} f_s \geq 0$ with $\iint_{\Omega \times \mathbf{R}^3} f_s \sqrt{\mu} = 0$ to the problem

$$v \cdot \nabla_x F_s = Q(F_s, F_s), \qquad F_s|_{\gamma_-} = \mu^\theta \int_{\gamma_+} F_s d\gamma,$$

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such that, for all $0 \leqq \zeta < \frac{1}{4+2\delta}, \ \beta > 4,$

$$||\langle v\rangle^\beta e^{\zeta|v|^2} f_s||_\infty + |\langle v\rangle^\beta e^{\zeta|v|^2} f_s|_\infty \lesssim \delta.$$

If $M\mu + \sqrt{\mu} g_s$ is an another solution with $\iint_{\Omega\times\mathbf{R}^3} g_s\sqrt{\mu} = 0$ such that, for $\beta > 4$

$$||\langle v\rangle^\beta g_s||_\infty + |\langle v\rangle^\beta g_s|_\infty \ll 1,$$

then $f_s \ \equiv \ g_s$.

Theorem : Continuity and Singularity

If $\theta(x)$ is continuous on $\partial\Omega$ then F_s is continuous away from $\mathfrak{D}$.
In particular, if Ω is convex then $\mathfrak{D} = \gamma_0$.
On the other hand, if Ω is not convex then we can construct a continuous function $\theta(x)$ on $\partial\Omega$ such that the corresponding solution F_s in not continuous.

Theorem : Dynamical Stability

Let $0 \leqq \zeta < \frac{1}{4+2\delta}$, $\beta > 4$. There exists $\varepsilon_0 > 0$, depends on δ_0, and $\lambda > 0$ such that if

$$||\langle v\rangle^{\beta} e^{\zeta|v|^2}[f(0) - f_s]||_{\infty} \leq \varepsilon_0$$

then there exists a unique non-negative dynamic solution $F(t) = \mu + f_s\sqrt{\mu} + f(t)\sqrt{\mu} \geq 0$ to the dynamical problem

$$\partial_t F + v \cdot \nabla_x F = Q(F, F), \qquad F(x, v) = \mu^{\theta} \int_{n(x)\cdot v>0} F \ n \cdot v$$

for $x \in \partial\Omega$ and $n(x) \cdot v < 0$ such that

$$||\langle v\rangle^{\beta} e^{\zeta|v|^2}[f(t) - f_s]||_{\infty} \lesssim e^{-\lambda t}||\langle v\rangle^{\beta} e^{\zeta|v|^2}[f(0) - f_s]||_{\infty}$$

Why $\delta-$Expansion ?

Fourier Law : a relation between the temperature and the heat flux

$$q_s = -\kappa(\theta_s)\partial_x \theta_s$$

for suitable positive smooth function κ.
Let F_s be the solution to the steady Boltzmann equation

$$\begin{aligned}
\theta_s(x) &= \frac{1}{3\rho_s}\int_{\mathbf{R}^3} |v-u_s|^2 F_s(x,v)dv \\
u_s(x) &= \frac{1}{\rho_s}\int_{\mathbf{R}^3} vF_s(x,v)dv \\
\rho_s(x) &= \int_{\mathbf{R}^3} F_s(x,v)dv \\
q_s(x) &= \frac{1}{2}\int_{\mathbf{R}^3}(v-u_s(x))|v-u(x)|^2 F_s(x,v)dv.
\end{aligned}$$

Purpose : See the first order characterization of F_s

What is $\delta-$Expansion ? : $\mu_\delta-$Expansion

Wall Temperature

$$\theta(x) \ = \ \theta_0 + \delta\vartheta(x), \quad |\vartheta(x)| \ \leq \ 1, \quad x \in \partial\Omega.$$

Wall Maxwellian

$$\mu_\delta(x,v) = \frac{1}{2\pi[\theta_0 + \delta\vartheta(x)]^2} \exp\left(-\frac{|v|^2}{2[\theta_0 + \delta\vartheta(x)]}\right)$$

Taylor Expansion in δ (μ_δ is analytic in δ)

$$\mu_\delta = \mu + \delta\mu_1 + \delta^2\mu_2 + \cdots + \delta^m\mu_m + \cdots$$

What is $\delta-$Expansion ? : $f_s \sim \delta f_1 + \delta^2 f_2 + \cdots$

Formal Expansion :

$$F_s = \mu + \sqrt{\mu}\Big\{\delta f_1 + \delta^2 f_2 + \cdots\Big\}$$
$$f_s = \delta f_1 + \delta^2 f_2 + \cdots$$

Plug in

$$v \cdot \nabla_x F_s = Q(F_s, F_s)$$

with Diffusive Bounadary Condition to get the linear equation for f_i (comparing the coefficients of power of δ)

Once we solve f_i, define the Remainder f_m^δ such that

$$f_s = \delta f_1 + \delta^2 f_2 + \cdots + \delta^m f_m^\delta$$

Theorem : $\delta-$Expansion

$\delta-$Expansion is valid !
There exist $f_1, f_2, \cdots, f_{m-1}$ and for all $i = 1, 2, \cdots m-1$

$$||\langle v\rangle^{\beta} e^{\zeta|v|^2} f_i||_\infty \lesssim 1$$

for all $0 \leqq \zeta < \frac{1}{4},\ \beta > 4$
and the remainder f_m^δ exits and

$$||\langle v\rangle^{\beta} e^{\zeta|v|^2} f_m^\delta||_\infty \lesssim 1$$

for all $0 \leqq \zeta < \frac{1}{4+2\delta},\ \beta > 4$

Theorem : Criterion for Fourier Law

Let $\Omega = [0,1]$. If the Fourier Law holds for $F_s = \mu + \sqrt{\mu} f_s$,

$$
\begin{aligned}
F_s &= \mu + \delta f_1 \sqrt{\mu} + O(\delta^2)\sqrt{\mu} \\
\theta_s &= \theta_0 + \delta \theta_1 + O(\delta^2)
\end{aligned}
$$

then

$$\theta_1(x) \quad \text{is a linear function on} \quad [0,1]$$

Prediction

From an available numeric simulation (Ohwada, Aoki, Sone, 1989)
θ_1 is not linear!

$$\Downarrow$$

Fourier Law is not valid at the kinetic regime !

Hydrodynamic Part : $\mathbf{P}f$

Linearized Boltzmann operator

- $Lf = -\frac{1}{\sqrt{\mu}}[Q(\mu, \sqrt{\mu}f) + Q(\sqrt{\mu}f, \mu)] = \nu(v)f - Kf$
- semi-positive : $\langle Lg, g\rangle \gtrsim ||\{\mathbf{I} - \mathbf{P}\}g||_\nu^2$
- kernel $=$ 'hydrodynamic part'

$$\mathbf{P}g \equiv \left\{ a_g(t,x) + v \cdot b_g(t,x) + \frac{|v|^2 - 3}{2} c_g(t,x) \right\} \sqrt{\mu}$$

Boltzmann equation $\Longrightarrow$ macroscopic equation for b_f

$$\Delta_x b_f = \partial_x^2 \{\mathbf{I} - \mathbf{P}\} f + \cdots$$

ellipticity in H^k $\Longrightarrow$ Guo:VMB(Invent.Math.2003), VPL(JAMS2011); Gressman-Strain:BE without angular cut-off(JAMS2011)

Difficulties with boundary conditions

$$\mathbf{P}f \equiv \left\{ a_f(t,x) + v \cdot b_f(t,x) + \frac{|v|^2 - 3}{2} c_f(t,x) \right\} \sqrt{\mu}$$

- $\mathbf{P}f$ and $\{\mathbf{I} - \mathbf{P}\}f$ do not make sense at the boundary
- no boundary condition for a_f, c_f, only $b_f \cdot n(x) = 0$ on $\partial\Omega$

Mathematical Framework : $L^2 - L^\infty$ Frame

Y.Guo : Initial Boundary Value Problem of BE, ARMA 2010

- L^2 Posivity : We Need A New Method !
- L^∞ Bound : We Need A New Method !

New L^2 Positivity Estimate

$$v \cdot \nabla_x f + Lf = g, \quad f_{\gamma_-} = P_\gamma f + r$$

$$\text{with} \quad \iint_{\Omega \times \mathbf{R}^3} f\sqrt{\mu} = 0 = \iint_{\Omega \times \mathbf{R}^3} g\sqrt{\mu} = \int_{n \cdot v < 0} r$$

$$\Longrightarrow \quad ||\mathbf{P}f||_\nu \leq M\{\underbrace{||(\mathbf{I} - \mathbf{P})f||_\nu + |(1 - P_\gamma)f|_{2,+}}_{Good \;\; Terms \;\; !}\} + \cdots$$

- weak formulation(Green's identity) + test functions
- constructive estimate with an explicit M
- dimension of $\Omega \; = \; 1, 2, 3$

New L^2 Positivity Estimate

Weak formulation (Green's identity)

$$\int_\gamma \psi f - \iint_{\Omega\times\mathbf{R}^3} v\cdot\nabla\psi f = -\iint_{\Omega\times\mathbf{R}^3}\psi L(\mathbf{I}-\mathbf{P})f + \iint_{\Omega\times\mathbf{R}^3}\psi g$$

$$\begin{aligned} \text{bulk} \quad & f &=& \quad \{a_f + v\cdot b_f + \frac{|v|^2-3}{2}c_f\}\sqrt{\mu} + (\mathbf{I}-\mathbf{P})f \\ \text{boundary} \quad & f_\gamma &=& \quad P_\gamma f + (1-P_\gamma) f \mathbf{1}_{\gamma_+} + r\mathbf{1}_{\gamma_-} \end{aligned}$$

New L^2 Positivity Estimate

Test functions

- for c_f : $\psi_c = (|v|^2 - \beta_c)\sqrt{\mu}\{v \cdot \nabla_x\}(-\Delta_0)^{-1} c_f$
  with $\int_{\mathbf{R}^3}(|v|^2 - \beta_c) v_i^2 \mu(v) dv = 0$
- for b_f :
  - $\psi_b^{i,j} = (v_i^2 - \beta_b)\sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_j$ for all $i, j = 1, 2, \cdots d$
    with $\int_{\mathbf{R}^3}(v_i^2 - \beta_b)\mu(v) dv = 0$, for all i
  - $\phi_b^{i,j} = v_i v_j |v|^2 \sqrt{\mu}\partial_j(-\Delta_0)^{-1}(b_f)_i$ for all $i \neq j$
- for a_f : $\psi_a = (|v|^2 - \beta_a)\{v \cdot \nabla_x\}(-\Delta_N)^{-1} a_f$
  with $\int_{\mathbf{R}^3}(|v|^2 - \beta_a)(\frac{|v|^2}{2} - \frac{3}{2})(v_i)^2 \mu(v) dv = 0$ for all i

Future

Thanks !