Ghost effect by curvature

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Ghost effect

The macroscopic equations for the hydrodynamic fields are derived from the Boltzmann equation in the limit when the Knudsen number $\mathcal{K}$ goes to 0.
von Karman relation between $\mathcal{K}$, $\mathcal{R}$ (Reynolds number) and $\mathcal{M}$ (Mach number):

$$\mathcal{M} \sim \mathcal{K}\mathcal{R}.$$

To keep the Reynolds number $\mathcal{R}$ finite, one has so take

$$\mathcal{M} = O(\mathcal{K}).$$

This and the assumption that temperature T and density ρ are constant at the lowest order in $\mathcal{K}$, as is well known, give the incompressible Navier-Stokes & Fourier equations as limit of the Boltzmann equation.
What about ρ and T non constant at the lowest order in $\mathcal{K}$?

Ghost effect

The formal answer can be obtained by expanding the solution to the Boltzmann equation in $\mathcal{K}$.

The result is somewhat unexpected: the macroscopic equations differ from the incompressible Navier-Stokes equations by the presence of a *thermal stress tensor* and $\mathrm{div} u \neq 0$.

Moreover, the equation for the temperature differs from the standard heat equation.

The responsible for this modification is the convective motion described by the velocity field u, which is $O(\mathcal{K})$, hence *invisible* in the limit $\mathcal{K} \to 0$, but affects the temperature which $O(1)$ in $\mathcal{K}$.

Ghost effect

For this reason Y. Sone called this phenomenon Ghost Effect.

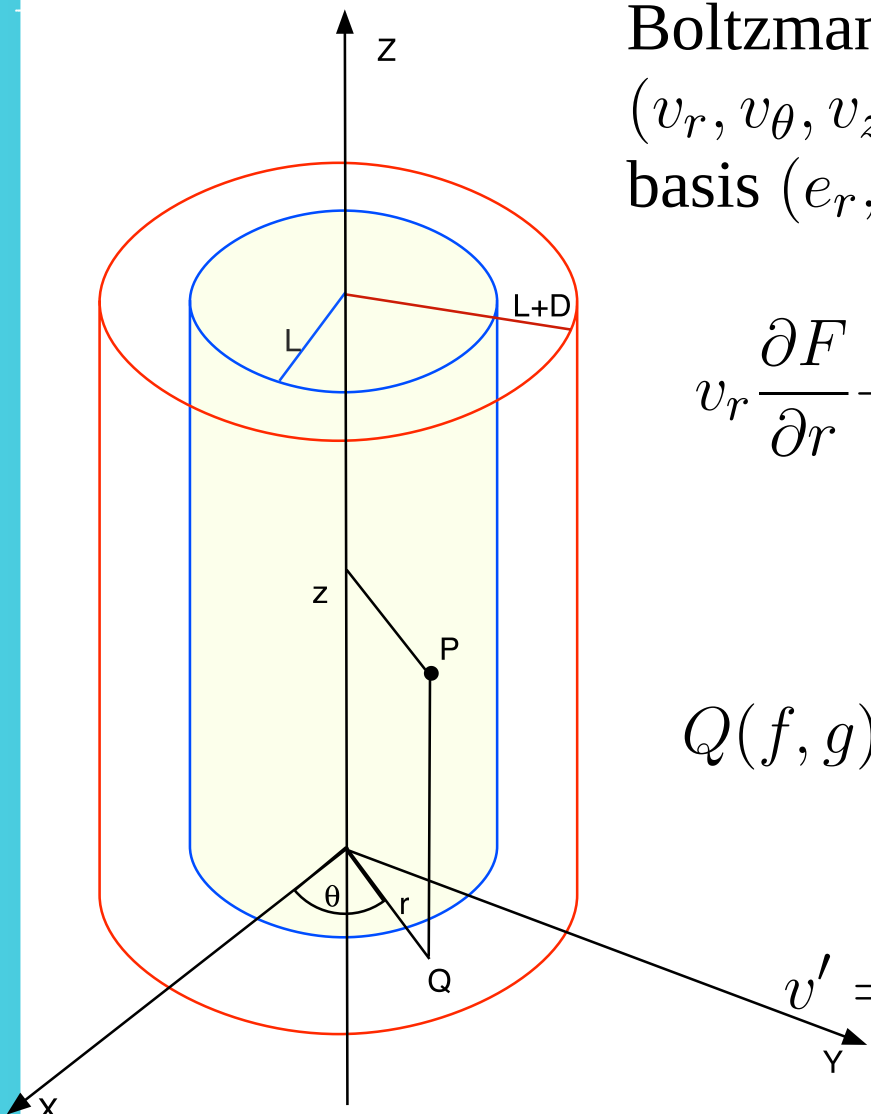
REMARK: If one starts from the standard compressible Navier-Stokes-Fourier equations, instead of the Boltzmann equation, and takes the limit $\mathcal{M} \to 0$ without assuming ρ and T constant, there is no ghost effect.

Ghost effect is a purely kinetic feature.

The term *ghost effect* is used by Sone to refer more generally to situations where the macroscopic limiting equations contain finite modifications due to the presence of quantities which are infinitesimal in the limit.

The ghost effect by curvature is an example of such situations.

Ghost effect by curvature



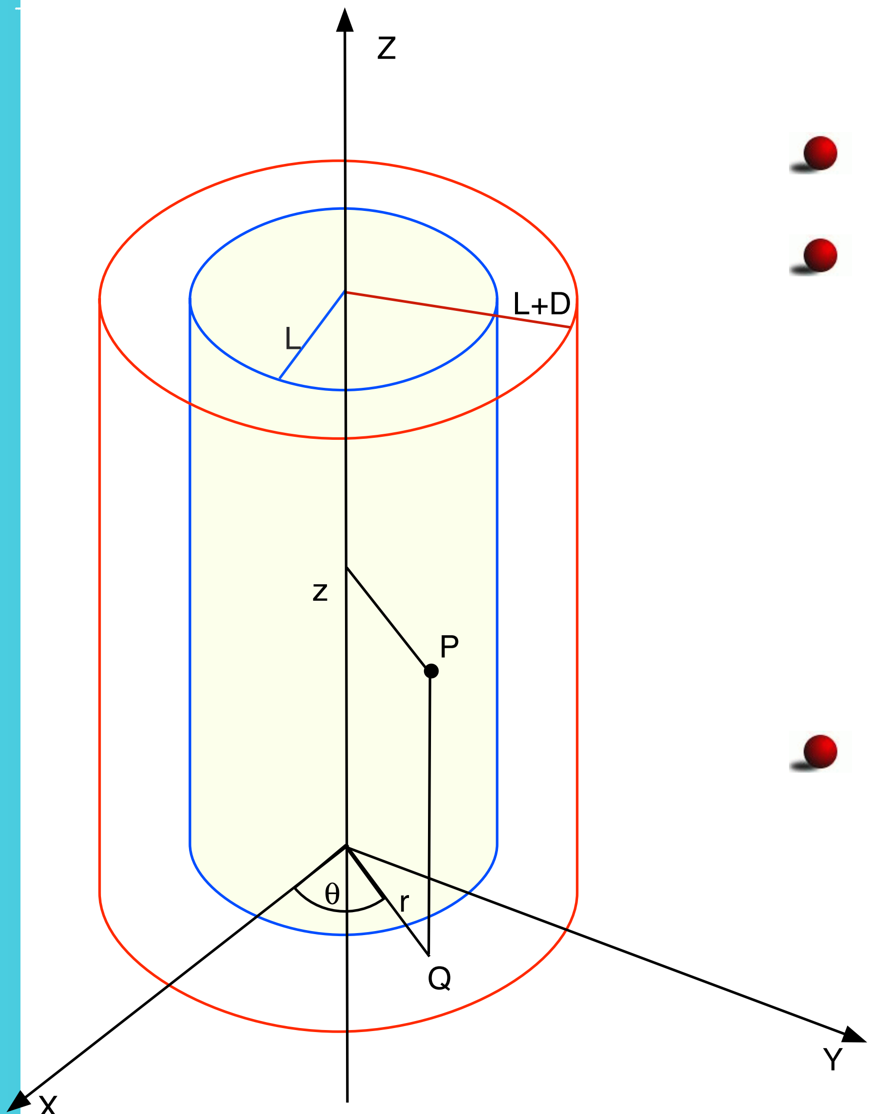
Boltzmann equation: $\varepsilon \sim \mathcal{K}$.
(v_r, v_θ, v_z) components of v in the basis (e_r, e_θ, e_z).

$$v_r \frac{\partial F}{\partial r} + \frac{v_\theta}{r}\frac{\partial F}{\partial \theta} + v_z \frac{\partial F}{\partial z} + \frac{v_\theta}{r}\left(v_\theta \frac{\partial F}{\partial v_r} - v_r \frac{\partial F}{\partial v_\theta} \right) = \frac{1}{\varepsilon} Q(F, F),$$

$$Q(f, g) = \frac{1}{2}\int_{\mathbb{R}^3} dv_* \int_{S_2} dn |n\cdot(v - v_*)| \times \{f'_* g' + f' g'_* - f_* g - g_* f\},$$

$$v' = v - nn\cdot(v - v_*), \quad v'_* = v_* + nn\cdot(v - v_*).$$

Ghost effect by curvature



- Fix $D = 1$.
- Change of variables

$$y = r - L, \quad x = -L\theta,$$

$$v_y = v_r, \quad v_x = -v_\theta.$$

- Scaling:

$$\frac{1}{L} = \frac{\varepsilon^2}{c^2}, \quad c > 0 \text{ specified later.}$$

Ghost effect by curvature

$$\zeta(y) v_x \frac{\partial F}{\partial x} + v_y \frac{\partial F}{\partial y} + v_z \frac{\partial F}{\partial z} + \frac{\varepsilon^2}{c^2} \zeta(y) v_x \left(v_x \frac{\partial F}{\partial v_y} - v_y \frac{\partial F}{\partial v_x} \right) = \frac{1}{\varepsilon} Q(F, F),$$

$$\zeta(y) = \frac{1}{1 + \frac{\varepsilon^2}{c^2} y}.$$

For simplicity, assume:

$$\frac{\partial F}{\partial x} \equiv 0.$$

Boundary conditions

Rotating cylinders at fixed temperatures:
Diffuse reflection boundary conditions.
Notation:

$$M(\rho, T, u; v) = \frac{\rho}{(2\pi T)^{3/2}} \mathrm{e}^{-\dfrac{|v-u|^2}{2T}}$$

On the boundaries:

$$u \cdot n \equiv u_y = 0.$$

$$\tilde{M}^{(i)}(v) = \sqrt{\frac{2\pi}{T^{(i)}}} M(1, T^{(i)}, u^{(i)}; v), \quad \int_{v_y>0} v_y \tilde{M} dv = 1, \quad i = 0, 1.$$

Boundary conditions

$$F(0,z,v) = \tilde{M}^{(0)}\alpha^{(0)}(F) \quad v_y > 0$$

$$F(1,z,v) = \tilde{M}^{(1)}\alpha^{(1)}(F) \quad v_y < 0,$$

$$\alpha^{(0)}(F) = -\int_{v_y<0} v_y F(0,z,v)dv, \quad \alpha^{(1)}(F) = \int_{v_y>0} v_y F(1,z,v)dv,$$

Zero mass flux at the boundary:

$$\int dv F(y,z,v)v_y = 0 \quad \text{for } y = 0,1.$$

Boundary conditions

The inner cylinder is at temperature $T = 1$ and rotates with speed $\mathcal{U}^{(0)}$; the outer cylinder is also at temperature $T = 1$ and rotates with speed $\mathcal{U}^{(1)}$.
We assume

$$\tilde{M}^{(0)} = \sqrt{2\pi} M(1, 1, (\mathcal{U}^{(0)}, 0, 0); v),$$

$$\tilde{M}^{(1)} = \sqrt{2\pi} M(1, 1, (\mathcal{U}^{(1)} 0, 0); v).$$

Low Mach number

Due to the special geometry, we do not need all the components of the velocity field of the order of the Knudsen number but only the y and z components. With the notation $\hat{v} = (v_y, v_z)$, we assume

$$\int dv \hat{v} F(y, z, v) = O(\varepsilon), \qquad \text{low Mach number}.$$

The radial flux $\int dv v_x F(y, z, v)$ is $O(1)$ in ε but we will need it to be small, say of order δ:

$$\delta \gg \varepsilon.$$

Therefore we set $\mathcal{U}^{(i)} = \delta U^{(i)}$, $i = 0, 1$, with $U^{(i)} = O(1)$ both in ε and in δ.

Local equilbrium

A current of order δ may produce temperature changes in the fluid of the same order and, consequently, density changes. Therefore one expects the solution at 0-th order in ε to be the *local equilibrium* with $\rho = 1 + \delta r$, $T = 1 + \delta\tau$:

$$M_\delta = M(1 + \delta r, 1 + \delta\tau, (\delta U, 0, 0); v),$$

where $\delta U(y, z)$ is the tangential component of the velocity field. We seek for the solution in the form

$$F(y, z, v) = M_\delta + \varepsilon[\Phi + \mathcal{R}]$$

with Φ a truncated expansion in ε and $\mathcal{R}$ a suitable remainder.

Expansion

$$\Phi = \sum_{n=1}^{N} \varepsilon^{n-1} F_n.$$

Plugging in the expansion one gets conditions on the F_n which, up to the first order in ε require

$$F_1 = M_\delta\Big[\rho_1 + \frac{u\cdot v}{1+\delta\tau} + \tau_1\Big(\frac{|v-Ue_x|^2 - 3(1+\delta\tau)}{2(1+\delta\tau)^2}\Big)\Big] + \mathcal{B}\cdot\hat{\nabla}\tau,$$

$$2Q(M_\delta, \mathcal{B}) \equiv \mathcal{L}_\delta \mathcal{B} = \tilde{\mathcal{B}},$$

$$\tilde{\mathcal{B}} = M_\delta \hat{v} \frac{|v-Ue_x|^2 - 5(1+\delta\tau)}{2(1+\delta\tau)^2},$$

Expansion

$$\hat{\nabla}\mathcal{P} = 0, \quad \hat{\nabla}\cdot[\hat{u}\rho] = 0, \qquad \hat{\nabla} \equiv (\partial_y, \partial_z)$$

$$\hat{u}\cdot\hat{\nabla}U = \rho^{-1}\hat{\nabla}\cdot(\eta\hat{\nabla}U),$$

$$\hat{u}\cdot\hat{\nabla}\hat{u} - \frac{\delta^2}{c^2}U^2 e_y = \rho^{-1}\Big[-\hat{\nabla}\mathcal{P}_2 + \hat{\nabla}\cdot(\eta\hat{\nabla}\hat{u}) + \hat{\nabla}\Sigma\Big]$$

$$\mathcal{P} = (1+\delta r)(1+\delta\tau) \quad \hat{u} = \rho^{-1}\int \hat{v}F_1 dv,$$

$$\Sigma = \hat{\nabla}\cdot\left(\frac{\delta^2}{\mathcal{P}}\big[\sigma_1\hat{\nabla}\tau\otimes\hat{\nabla}\tau + \sigma_2\hat{\nabla}U\otimes\hat{\nabla}U\big]\right).$$

with η, σ_1, σ_2 smooth functions of T.

Expansion

The equation for the temperature τ is

$$\frac{5}{2}\hat{u}\cdot\hat{\nabla}\tau = \rho^{-1}\Big[\hat{\nabla}(\kappa\hat{\nabla}\tau) + \delta\eta|\hat{\nabla}U|^2\Big],$$

with κ a smooth function of T.

Limit for $\delta \to 0$

The macroscopic equations become much simpler in the limit $\delta \to 0$. In order to keep the correction term in the equation

$$\hat{u} \cdot \hat{\nabla} \hat{u} - \frac{\delta^2}{c^2} U^2 e_y = \rho^{-1} \Big[- \hat{\nabla} \mathcal{P}_2 + \hat{\nabla} \cdot (\eta \hat{\nabla} \hat{u}) + \hat{\nabla} \Sigma \Big],$$

we set (with $\frac{1}{L} = \frac{\varepsilon^2}{c^2}$)

$$c = \delta C$$

with C and $O(1)$ constant related to the curvature.
Hence the scaling becomes

$$\frac{1}{L} = \frac{\varepsilon^2}{C^2 \delta^2}.$$

Limiting equations

$$\hat{\nabla} \cdot \hat{u} = 0,$$
$$\hat{u} \cdot \hat{\nabla} U = \eta_0 \hat{\Delta} U,$$
$$\hat{u} \cdot \hat{\nabla} \hat{u} + \hat{\nabla} \mathcal{P}_2 - \frac{1}{C^2} U^2 e_y = \eta_0 \hat{\Delta} \hat{u},$$
$$\frac{5}{2} \hat{u} \cdot \hat{\nabla} \tau = \kappa_0 \hat{\Delta} \tau,$$

with $\eta_0 = \eta(1)$, $\kappa_0 = \kappa(1)$. $\mathcal{P}_2$ second order pressure.

Boundary conditions

We consider now the following boundary conditions:

$$U(0,z) = U^{(0)}, \quad U(1,z) = U^{(1)},$$

$$\hat{u}(0,z) = \hat{u}(1,z) = 0,$$

$$\tau(0,z) = \tau(1,z) = 0.$$

Bifurcation $\delta = 0$

Set $\beta = U^{(0)} - U^{(1)}$. Laminar solution:

$$U_\ell(y) = U^{(1)} + \beta(1-y), \quad \hat{u}_\ell = 0, \quad \tau_\ell = 0.$$

There is β_c such that, for $\beta > \beta_c$ (but close to β_c) the solution bifurcate into two non laminar solutions $U_\pm(y,z)$, $\hat{u}_\pm(y,z)$, with $\tau = 0$. $\|U_\pm - U_\ell\| = O(\beta - \beta_c)$, $\|\hat{u}_\pm\| = O(\beta - \beta_c)$.

No such a bifurcation in planar Couette flow.

The term $C^{-2}U^2 e_y$ is responsible for the bifurcation. It is due the curvature which on the other hand goes to 0 in this scaling limit. Ghost effect.

Bifurcation $\delta = 0$

Linear analysis: Sone-Doi [SD]

Theorem both for linear and non linear bifurcation in Arkeryd-R.E.-Marra-Nouri [AEMN] in preparation.

Bifurcation $\delta \neq 0$

When $\delta \neq 0$ the system is slightly compressible.

Bifurcation persists: there are solutions $(U^\delta_\pm, \hat{u}^\delta_\pm, \tau^\delta_\pm)$ such that

$$\|(U^\delta_\pm, \hat{u}^\delta_\pm, \tau^\delta_\pm) - (U_\pm, \hat{u}_\pm, 0)\| = O(\delta).$$

Perturbation argument. Extra complications due to the (small) compressibility.

Theorem proved in [AEMN] in H^s-norms.

Boundary layer

The solution is written as

$$F = M_\delta + \varepsilon[\Phi + \mathcal{R}].$$

Φ computed with the expansion does not satisfy the diffuse reflection boundary conditions.
We need boundary layer corrections:

$$\Phi = \sum_{n=1}^{N} \varepsilon^{n-1}\Phi_n, \quad \Phi_n = F_n + b_n^{(0)} + b_n^{(1)}.$$

The boundary correction $b_n^{(i)}$, depending on $\varepsilon^{-1}y$, solve a modified Milne problem.

Milne problem

Set $Y^{(0)} = \varepsilon^{-1}y$, $Y^{(1)} = \varepsilon^{-1}(1-y)$

$$v_y \frac{\partial b_n^{(i)}}{\partial Y^{(i)}} + \frac{\varepsilon^3}{C^2\delta^2}\zeta\mathcal{N}(b_n^{(i)}) = \mathcal{L}^{(i)} b_n^{(i)} + s_n \quad \text{in } (0, +\infty).$$

with $\mathcal{L}^{(i)}g = 2Q(M^{(i)}, g)$, $M^{(i)} = M(1, 1, (\delta U^{(i)}, 0, 0); v)$.

$$\mathcal{N}(f) = v_x \left(v_x \frac{\partial f}{\partial v_y} - v_y \frac{\partial f}{\partial v_x} \right),$$

with suitable prescribed incoming data in $Y^{(i)} = 0$ and source term s.

Adding some cutoff and regularization we have proved *existence, regularity (away from the boundary) and exponential decay of the solution* in [AEMN1].

Expansion

Let M_δ be computed on one of the stationary solutions to the hydrodynamical equations with $\delta > 0$, namely perturbations of the laminar or the bifurcating solution, and Φ the corresponding bulk & boundary layer expansion.

It can be proved that M_δ and Φ have the smoothness and boundedness properties needed in the rest of the argument. Moreover (set $M = M_{\delta=0}$)

$$M_\delta = M(1 + O(\delta)), \quad \Phi = O(\delta).$$

The stationary solution to the Boltzmann equation is written as:

$$F = M_\delta + \varepsilon[\Phi + \mathcal{R}].$$

Remainder

Equation for $\mathcal{R}$:

$$v_y \frac{\partial \mathcal{R}}{\partial y} + \frac{\varepsilon^2}{\delta^2 C^2} \zeta(y) \mathcal{N}(\mathcal{R}) = \frac{1}{\varepsilon} \Big[\mathcal{L}_\delta \mathcal{R} + 2Q(\varepsilon \Phi, \mathcal{R}) \Big] + \varepsilon Q(\mathcal{R}, \mathcal{R}) + \mathcal{A},$$

$$\mathcal{L}_\delta \mathcal{R} = 2Q(M_\delta, \mathcal{R}),$$

The inhogeneous term $\mathcal{A}$ is expressed in terms of F_n's and b_n's.
It's size is ε^m with m depending on the number of terms in the expansion N.
Moreover it is bounded in the relevant norms.

Boundary conditions for $\mathcal{R}$

$$\mathcal{R}(0,z,v) = \alpha^{(0)}(\mathcal{R} + \varepsilon^{-1}\Psi^{(0)})\tilde{M}^{(0)} - \frac{1}{\varepsilon}\Psi^{(0)}(z,v), \quad v_y > 0$$

$$\mathcal{R}(1,z,v) = \alpha^{(1)}(\mathcal{R} + \varepsilon^{-1}\Psi^{(1)})\tilde{M}^{(1)} - \frac{1}{\varepsilon}\Psi^{(1)}(z,v), \quad v_y < 0,$$

where the inhomegeneous term $\Psi^{(i)}$ depends on the b_n's and are exponentially small in ε in the relevant norms.

Green Identity

Given H and K, consider the equation

$$v \cdot \nabla_x F - \frac{1}{\varepsilon}\mathcal{L}_\delta F - H = 0, \qquad \text{in } \Omega \times \mathbb{R}^3,$$

$$F(x,v) = K \quad (x,v) \in \gamma_- = \{(x,v) \in \partial\Omega \times \mathbb{R}^3 \,|\, n(x) \cdot v < 0\}$$

$n(x)$ exterior normal to $\partial\Omega$ in x.
Given a positive weight function $q(x,v)$, multiply the equation by qF and integrate on (x,v).One gets ($(f,g)_q := (f,\, qg)_{L^2}$)

$$-\frac{1}{\varepsilon}(F, \mathcal{L}_\delta F)_q + \frac{1}{2}\int_{\gamma_+} v \cdot n q F^2 = (F,H)_q + \frac{1}{2}\int_{\gamma_-} |v \cdot n| q K^2 + \frac{1}{2}\int_{\Omega \times \mathbb{R}^3} F^2 v \cdot \nabla_x q$$

Green Identity

Such an identity is useful if the quadratic form $(F, \mathcal{L}_\delta F)_q$ is non positive.

This is true if one chooses $q = M_\delta^{-1}$. M_δ local equilibrium.

But then $v \cdot \nabla_x q$ is a polynomial of degree 3 in v and the r.h.s of the Green inequality <u>cannot</u> be controlled directly in terms of the l.h.s. [Caflisch].

On the other hand $\mathcal{L}_\delta = \mathcal{L} + O(\delta)$, with $\mathcal{L} = \mathcal{L}_{\delta=0} = 2Q(M, \cdot)$.

With $q = M^{-1}$ we have

$$-(F, \mathcal{L}F)_{M^{-1}} \geq c(P^\perp F, \nu P^\perp F)_{M^{-1}},$$

with c a positive constant, P the projector onto the the null space of $\mathcal{L}$, $P^\perp \equiv 1 - P$ and ν a known function of v.

Spectral inequality

If we could neglect $O(\delta)$ in the relation $\mathcal{L}_\delta = \mathcal{L} + O(\delta)$, then we could use the Green inequality to control $P^\perp F$ in L^2.
But we cannot neglect $O(\delta)$ because of the factor ε^{-1}: In the Green identity we would have $\varepsilon^{-1} O(\delta)$.
Set $W = \delta^{-1}(M_\delta - M) = O(1)$. Then

$$\frac{1}{\varepsilon}(F, \mathcal{L}_\delta F) = \frac{1}{\varepsilon}(F, \mathcal{L} F) + 2\frac{\delta}{\varepsilon}(F, Q(W, F))$$

$$\frac{\delta}{\varepsilon}|(F, Q(W, P^\perp F)_{M^{-1}}| \le \frac{\delta}{\varepsilon}\|W\|_\infty (P^\perp F, \nu P^\perp F)_{M^{-1}}$$

$$\ll \frac{1}{\varepsilon} c (P^\perp F, \nu P^\perp F)_{M^{-1}}.$$

Spectral inequality

The dangerous part of the $\varepsilon^{-1}O(\delta)$ term comes from

$$\varepsilon^{-1}\delta(F, Q(W, PF))_{M^{-1}}$$

which may be bigger than $|(F, \mathcal{L}F)_{M^{-1}}|$.

We define

$$-\mathcal{L}^W F = \mathcal{L}F + 2\delta Q(W + \delta^{-1}\Phi, PF).$$

Theorem[AEMN2]: *There are* $\delta_0 > 0$ *and* $c > 0$ *such that, if* $\delta < \delta_0$ *then*

$$-(F, \mathcal{L}^W F)_{M^{-1}} \geq c((1 - P^W)F, \nu(1 - P^W)F)_{M^{-1}},$$

where $P^W = P + O(\delta)$ *is the projector onto the null space of* $\mathcal{L}^W$*.*

Green inequality

Hence, the remainder $\mathcal{R}$ satisfies the inequality

$$\frac{c}{\varepsilon}\|(1-P^W)\mathcal{R},(1-P^W)\mathcal{R}\|^2_{\nu M^{-1}} + \frac{1}{2}\int_{\gamma_+} v\cdot n M^{-1}\mathcal{R}^2 \leq (\mathcal{R},H)_{M^{-1}} + \frac{1}{2}\int_{\gamma_-}|v\cdot n|M^{-1}K^2$$

$$H = -\frac{\varepsilon^2}{\delta^2 C^2}\zeta(y)\mathcal{N}(\mathcal{R}) + \varepsilon Q(\mathcal{R},\mathcal{R}) + \mathcal{A},$$

$$K(0,z,v) = \alpha^{(0)}(\mathcal{R} + \frac{1}{\varepsilon}\psi^{(0)})\tilde{M}^{(0)} - \frac{1}{\varepsilon}\Psi^{(0)}(z,v), \quad v_y > 0$$

$$K(1,z,v) = \alpha^{(1)}(\mathcal{R} + \frac{1}{\varepsilon}\psi^{(1)})\tilde{M}^{(1)} - \frac{1}{\varepsilon}\Psi^{(1)}(z,v), \quad v_y < 0$$

Boundary inequality

When $\tilde{M}^{(0)} = \tilde{M}^{(1)} = \sqrt{2\pi}M$, then

$$\frac{1}{2}\int_{\partial\Omega\times\mathbb{R}^3} v_y M^{-1}\mathcal{R}^2 \geq 0$$

by Darrozés-Guiraud inequality. In this case $\tilde{M}^{(0)} \neq \tilde{M}^{(1)}$ and one needs to estimate it. We can show that

$$\left|\int_{\partial\Omega\times\mathbb{R}^3} v_y M^{-1}\mathcal{R}^2\right| \leq C\left[\|P^W\mathcal{R}\|^2 + \frac{\delta^2}{\varepsilon^2}\|(1-P^W)\mathcal{R}\|^2\right]$$

This is useful for

$$\frac{\delta^2}{\varepsilon^2} = \frac{\gamma}{\varepsilon}, \quad \gamma \ll 1.$$

Estimate for $\mathcal{N}(\mathcal{R})$

Simple algebra shows that

$$\int_{\mathbb{R}^3} dv M^{-1} \mathcal{R} \mathcal{N}(\mathcal{R}) = \frac{1}{2} \int_{\mathbb{R}^3} dv M^{-1} \mathcal{R}^2 v_y.$$

Thus

$$\partial_y \int_{\mathbb{R}^3} dv M^{-1} \mathcal{R}^2 v_y + \frac{\varepsilon^2}{\delta^2 C^2} \zeta(y) \int_{\mathbb{R}^3} dv M^{-1} \mathcal{R} \mathcal{N}(\mathcal{R})$$
$$= \frac{1}{\sigma(y)} \partial_y \left[\sigma(y) \int_{\mathbb{R}^3} dv M^{-1} \mathcal{R}^2 v_y \right].$$

for a suitable function σ. Using this one can take care of the contribution from the *centrifugal force*.

Estimate of $\|(1-P^W)\mathcal{R}\|_{\nu M^{-1}}$

Set $\tilde{\mathcal{A}} = \mathcal{A} + \varepsilon Q(\mathcal{R}, \mathcal{R})$ and $\|\ \cdot\ \| \equiv \|\ \cdot\ \|^2_{\nu M^{-1}}$.
Collecting above arguments one proves that, if ε and γ are sufficiently small, then for any $\eta > 0$

$$\|(1-P^W)\mathcal{R}\|^2 \leq (\eta+\gamma)\varepsilon\|P^W\mathcal{R}\|^2 + \frac{1}{\varepsilon\eta}\|P^W\tilde{\mathcal{A}}\|^2 + \varepsilon\|(1-P)^W\tilde{\mathcal{A}}\|^2 + \frac{1}{\eta}O(e^{-\frac{1}{\varepsilon}}).$$

To conclude the argument we need:

1) An estimate of $\|P^W\mathcal{R}\|$;

2) Control of the nonlinearity.

Estimate of $\|P\mathcal{R}\|$

First consider the 1-d case. It is not obvious but true that

$$\int_{\mathbb{R}^3} dv \mathcal{R}(1, v) v_y = 0.$$

Then one can prove that, if $\int_{\mathbb{R}^3} dv \tilde{\mathcal{A}} = 0$, then

$$\|P^W \mathcal{R}\|^2 \leq c\big(\|(I - P^W)\tilde{\mathcal{A}}\|^2 + \frac{1}{\varepsilon^2}\|P^W \tilde{\mathcal{A}}\|^2 + O(e^{-\frac{1}{\varepsilon}}\big).$$

Conclusion for $d = 1$

The control of the non linear collision term is based on getting $L^2(\mathbb{R}^3_v; L^\infty_x(\Omega))$ estimates using characteristics. Then an iterative procedure implies existence and boundedness of the remainder for $d = 1$:

Theorem[AEMN1]: *Assume $\mathcal{A} = O(\varepsilon^4)$ and ε and $\delta^2\varepsilon^{-1}$ small. Then there is a unique solution $\mathcal{R}$ and*

$$\|\mathcal{R}\|_\infty = O(\varepsilon^{3/2}).$$

Therefore, corresponding to the laminar solution of the macroscopic equations there is an isolated L^2-solution F to the Boltzmann equation such that,

$$\|M^{-1}[F - M_\delta]\|_{L^2([0,1]\times\mathbb{R}^3)} \leq c\varepsilon.$$

$d = 2$

Since $U_{\pm}(y,z)$, $\hat{u}_{\pm}(y,z)$ are $O(\beta - \beta_c)$, we use perturbation arguments of the 1-d case. Following a precedure similar to the one employed in [AEMN2] for the Benard problem, we can prove the following:

Theorem[AEMN - in preparation]: *Let $\beta - \beta_c$ be positive and small (independently of ε and δ). Let M_δ be the Maxwellian with parameters given by one of the δ-perturbed bifurcating solution. Then, for ε and $\delta^2\varepsilon^{-1}$ small, there is an isolated L^2-solution F to the stationary Boltzmann equation such that*

$$\|M^{-1}[F - M_\delta]\|_{L^2([0,1]\times\mathbb{R}^3)} \leq c\varepsilon.$$

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