Large time behavior of coagulation-fragmentation equations with degenerate diffusion

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A strange decay rate

Convergence towards equilibrium in

$$||f(t) - f_{eq}|| \leq Cst\, e^{-Cst\,(\log t)^{\beta}},$$

for all $\beta < 2$.

Better than any power; Worse than any $Cst\ \exp(-Cst\, t^{\gamma})$, with $\gamma > 0$.

Ingredients

- √ **Entropy (Entropy dissipation) method**
- √ **Almost exponential convergence to equilibrium**
- √ **Tracking explicitly the evolution of moments**

Entropy method for large time behavior

Abstract equation :

$$\partial_t f = A\, f.$$

We suppose that there exists a (bounded below) Lyapounov functional $H := H(f)$ (entropy) and a functional $D := D(f)$ (entropy dissipation) such that

$$\partial_t H(f) = -D(f) \leq 0,$$

and

$$D(f) = 0 \quad \Longleftrightarrow \quad A\, f = 0 \quad \Longleftrightarrow \quad f = f_{eq},$$

where f_{eq} is a given function.

De La Salle's Principle

Then, one can often prove that

1. the decreasing function $t \mapsto H(f(t))$ converges toward its minimum $H(f_{eq})$,
2. $t \mapsto D(f(t))$ converges toward 0,
3. $$\lim_{t \to +\infty} f(t) = f_{eq}$$

   in a convenient topology.

Explicit rate of convergence toward equilibrium

One looks for functional inequalities like

$$D(f) \geq Cst\,(H(f) - H(f_{eq})).$$

Then, using the Lyapounov functional, we get the differential inequality

$$\partial_t (H(f) - H(f_{eq})) \leq -Cst\,(H(f) - H(f_{eq})),$$

and Gronwall's lemma ensures that

$$H(f) - H(f_{eq}) \leq Cst\, e^{-Cst\,t}.$$

Finally, if H has good properties of coercivity, we obtain

$$||f - f_{eq}|| \leq Cst\, e^{-Cst\,t}.$$

Coagulation-fragmentation

Aizenmann-Bak model

$$\partial_t f(t,y) = Q_{AB}(f)(t,y).$$

1. Break-ups of clusters of size y' larger than y contribute to create clusters of size y:

$$Q^+_{frag}(f)(t,y) := 2\int_y^\infty f(t,y')\,dy'.$$

2. Break-up of polymers of size y reduces its concentration:

$$Q^-_{frag}(f)(t,y) := y\,f(t,y).$$

1. Coalescence of clusters of size $y' \leq y$ and $y - y'$ results into clusters of size y:

$$Q^+_{coag}(f)(t,y) := \int_0^y f(t, y-y') f(t,y')\, dy'.$$

2. Polymerization of clusters of size y with other clusters of size y' produces a loss in its concentration:

$$Q^-_{coag}(f)(t,y) := 2f(t,y) \int_0^\infty f(t,y')\, dy'.$$

Aizenmann-Bak kernel :

$$Q_{AB}(f)(t,y) = Q^+_{frag}(f)(t,y) - Q^-_{frag}(f)(t,y) + Q^+_{coag}(f)(t,y) - Q^-_{coag}(f)(t,y)\,.$$

Entropy structure for Aizenmann-Bak model

Entropy:

$$H_{AB}(f)(t) = \int_0^\infty \Big(f(t,y)\, \log f(t,y) - f(t,y) \Big)\, dy\,,$$

Entropy dissipation:

$$D_{AB}(f)(t) = \int_0^\infty \int_0^\infty \Big(f(t, y+y') - f(t,y)\, f(t,y') \Big)$$

$$\times \Big(\log[f(t,y+y')] - \log[f(t,y)\, f(t,y')] \Big)\, dy'\, dy \;\geq 0\,,$$

Entropy relation:

$$\frac{d}{dt} H_{AB}(f)(t) = \int_0^\infty Q_{AB}(f,f)(t,y)\, \log f(t,y)\, dy = -D_{AB}(f)(t).$$

Case of equality in the entropy structure for Aizenmann-Bak kernel

$$D_{AB}(f) = 0 \qquad \Longleftrightarrow \qquad \forall y \geq 0, \quad Q_{AB}(f)(y) = 0$$

$$\Longleftrightarrow \qquad \forall y, y' \geq 0, \quad f(y + y') = f(y)\, f(y')$$

$$\Longleftrightarrow \qquad \exists c > 0, \qquad f(y) = \exp(-c\, y)$$

Analogous to the second part (case of equality) of Boltzmann H-theorem.

Entropy/Entropy Dissipation estimate for AB model

Theorem (M. Aizenmann, T. Bak): For any $f := f(y) \geq 0$ such that $N_f \geq N_* > 0$,

$$D_{AB}(f) \geq Cst(N_*)\ (H_{AB}(f) - H(M_{AB}(f)))\,,$$

where

$$M_{AB}(f)(y) = e^{-\frac{y}{\sqrt{N_f}}}, \qquad N_f = \int_0^\infty f(y)\, y\, dy.$$

Main tool: Elementary convexity inequalities

Analogous result for a discrete model : P.-E. Jabin, B. Niethammer

Explicit rate of convergence toward equilibrium (AB)

Theorem (M. Aizenmann, T. Bak): Let $f_{in} := f_{in}(y) \geq 0$ be an initial datum such that

$$\int_0^\infty f_{in}(y)\,\left(1 + y + |\log f_{in}(y)|\right)\,dy < +\infty.$$

Then there exists a unique solution to the Aizenmann-Bak equation

$$\partial_t f(t,y) = Q_{AB}(f)(t,y), \qquad f(0,y) = f_{in}(y),$$

such that

$$N(f(t,\cdot)) = \int_0^\infty f(t,y)\,y\,dy = \int_0^\infty f(0,y)\,y\,dy = N(f_{in}).$$

Moreover this solution satisfies (for some explicit $C_1, C_2 > 0$)

$$\left|\left| f(t,y) - \exp\left(- \frac{y}{\sqrt{N(f_{in})}} \right) \right|\right|_{L^1(\mathbb{R}_+)} \leq C_1\, e^{-C_2\, t}.$$

Inhomogeneous Aizenmann-Bak equation

New variables for the unknown:

$$f(t, y) \to f(t, x, y).$$

Equation (for some $a(y) \geq 0$):

$$\partial_t f(t, x, y) - a(y)\, \Delta_x f(t, x, y) = Q_{AB}(f)(t, x, y).$$

Homogeneous Neumann boundary conditions:

$$\forall x \in \partial\Omega, \qquad \nabla_x f(t, x, y) \cdot n(x) = 0.$$

Initial datum:

$$f(0, x, y) = f_{in}(x, y).$$

Entropy structure for the Inhomogeneous AB equation

Entropy:

$$H_{IAB}(f)(t) = \int_\Omega \int_0^\infty \Big(f(t,x,y)\, \log f(t,x,y) - \; f(t,x,y) \Big)\, dydx\,.$$

Entropy relation:

$$\partial_t H_{IAB}(f) = -(D_1(f) + D_2(f)).$$

Entropy dissipation:

$$D_1(f)(t) = \int_\Omega \int_0^\infty a(y)\, \frac{|\nabla_x f(t,x,y)|^2}{f(t,x,y)}\, dydx,$$

$$D_2(f)(t) = \int_\Omega \int_0^\infty \int_0^\infty \Big(f(t,x,y+y') - f(t,x,y)\, f(t,x,y') \Big)$$

$$\times \Big(\log[f(t,x,y+y')] - \log[f(t,x,y)\, f(t,x,y')] \Big) dy'\, dydx\,.$$

Case of equality in the entropy structure of the inhomogeneous AB model

$$D_1(f) + D_2(f) = 0$$

$$\Longleftrightarrow \qquad f(x,y) = e^{-c(x)\,y} \qquad \text{and} \qquad \nabla_x c(x) = 0$$

$$\Longleftrightarrow \qquad f(x,y) = e^{-c\,y}.$$

Entropy/entropy dissipation estimate for the inhomogeneous AB equation

Proposition (J. Carrillo, LD, K. Fellner): Let $f := f(x,y) \geq 0$ be such that

$$M(x) := \int_0^\infty f(x,y)\,dy \geq M_* > 0, \qquad N(x) := \int_0^\infty f(x,y)\,y\,dy \geq N_* > 0.$$

Then

$$D_1(f) + D_2(f) \geq \frac{Cst(M_*, N_*, \inf a, \sup a)}{||M||_{L^\infty(\Omega)} \log(||M||_{L^\infty(\Omega)})}\,(H_{IAB}(f) - H_{IAB}(f_{eq})),$$

with

$$f_{eq}(x,y) = e^{-y\sqrt{\frac{|\Omega|}{\int N(x)\,dx}}}.$$

Large time behavior of the inhomogeneous AB equation

Theorem (J. Carrillo, LD, K. Fellner): Let $0 < a_* \leq a(y) \leq a^*$, and $f_{in} := f_{in}(x,y) \geq 0$ be an initial datum such that

$$\int_0^1 \int_0^\infty f_{in}(x,y)\,(1 + y + |\log f_{in}(x,y)|)\, dy dx < +\infty.$$

According to a theorem by Ph. Laurençot, S. Mischler, there exists a unique solution $f := f(t,x,y) \geq 0$ to the spatially inhomogeneous Aizenmann-Bak equation with Neumann BC and initial datum f_{in}.
Moreover, for c given by the conservation of total mass and for all $q \in \mathbb{N}$,

$$\int_0^\infty (1+y)^q \,||f(t,\cdot,y) - e^{-c\,y}||_{L^\infty(]0,1[)}\, dy \leq C_1\, e^{-C_2\, t},$$

where $C_1, C_2 > 0$ are explictly computable in terms of a^*, a_*, q, f_{in}.

Estimates used in the proof

1. Bounds from above using the entropy dissipation

$$M \in (L^1 + L^\infty)([0, +\infty[; L^\infty(]0, 1[)),$$

2. Bounds from below using the heat kernel and the conservation of total mass

$$M(t, x) \geq M_* > 0, \qquad N(t, x) \geq N_* > 0,$$

3. Cziszar-Kullback-Pinsker inequality,
4. Smoothness estimates using the heat kernel : for all $q \in \mathbb{N}$,

$$\int_0^\infty (1 + y)^q \, ||f(t, \cdot, y)||_{H^1(]0,1[)} \, dy \leq Pol(t).$$

Almost exponential convergence to equilibrium

Sometimes, the functional inequality

$$D(f) \geq C\,(H(f) - H(f_{eq}))$$

cannot be proven, and one has instead

$$D(f) \geq C\,A^{-1}\,(H(f) - H(f_{eq})) - C_p\,A^{-(p+1)}$$

for some or all $p > 0$, and all $A > 0$.

Together with “slowly growing a priori coefficients”: G. Toscani, C. Villani

Almost exponential convergence to equilibrium

$$D(f) \geq C\, A^{-1}\,(H(f) - H(f_{eq})) - C_p\, A^{-(p+1)}$$

for some or all $p > 0$, and all $A > 0$.

Then by interpolation,

$$D(f) \geq C_p\,(H(f) - H(f_{eq}))^{(p+1)/p},$$

and we get the differential inequality

$$\partial_t (H(f) - H(f_{eq})) \leq -C_p\,(H(f) - H(f_{eq}))^{(p+1)/p},$$

so that thanks to Gronwall's lemma,

$$H(f)(t) - H(f_{eq}) \leq C_p\,(1+t)^{-p}.$$

Strange rate of convergence toward equilibrium

If $C_p = 2^{2p^2}$, then

$$2^{-2p^2}\, t^p \, [H(f)(t) - H(f_{eq})] \leq 1.$$

Taking $p = (\log t)^{1-\varepsilon}$ for some $\varepsilon > 0$, we get

$$\frac{e^{(\log t)^{2-\varepsilon}}}{e^{2\ \log 2\ (\log t)^{2-2\varepsilon}}}\,(H(f)(t) - H(f_{eq})) \leq 1,$$

so that

$$H(f)(t) - H(f_{eq}) \leq Cst\, e^{-Cst\ (\log t)^{2-\varepsilon}}.$$

Large time behavior of the degenerate inhomogeneous AB equation

Theorem (LD, K. Fellner): Let $0 < \frac{a_*}{1+y} \le a(y) \le a^*$, and $f_{in} := f_{in}(x,y) \ge 0$ be an initial datum such that

$$\int_0^1 \int_0^\infty f_{in}(x,y)\,(1+y+|\log f_{in}(x,y)|)\,dydx < +\infty.$$

According to a theorem by Ph. Laurençot, S. Mischler, there exists a unique solution $f := f(t,x,y) \ge 0$ to the spatially inhomogeneous Aizenmann-Bak equation with Neumann BC and initial datum f_{in}.
Moreover, for c given by the conservation of total mass and for all $\beta < 2$,

$$||f(t,\cdot,*) - e^{-c\,*}||_{L^1(]0,1[\times\mathbb{R}_+)} \le C_\beta\, e^{-(\log t)^\beta}.$$

where $C_\beta > 0$ is explicitly computable in terms of a^*, a_*, β, f_{in}.

Estimates used in the proof

1. Bounds from above using the entropy dissipation

$$M \in (L^1 + L^\infty)([0, +\infty[; L^\infty(]0, 1[),$$

2. Bounds from below using the heat kernel and the conservation of total mass

$$M(t, x) \geq M_* > 0,$$

3. Cziszar-Kullback-Pinsker inequality,
4. Estimates on the moments $\mathcal{M}_p(f)(t) := \int_0^1 \int_0^\infty y^p \, f(t, x, y) \, dy dx$.

Estimates used in the proof: cutoff for large y

We define $M_A(t,x) := \int_0^A f(t,x,y)\,dy$ and $\overline{M} = \int M\,dx$. Then

$$\|M-\overline{M}\|^2_{L^2_x} \le 2\|M_A-\overline{M_A}\|^2_{L^2_x} + \frac{4}{A^{2p}}\int_0^1\left(\int_0^\infty y^p f(t,x,y)\,dy\right)^2 dx$$
$$\le 2\|M_A-\overline{M_A}\|^2_{L^2_x} + \frac{4}{A^{2p}}\|M\|_{L^\infty_x}\mathcal{M}_{2p}\,,$$

$$\|M_A-\overline{M_A}\|^2_{L^2_x} \le P(\Omega)\int_0^1\left|\nabla_x\int_0^A f\,dy\right|^2 dx$$
$$\le P(\Omega)\int_0^1\left|\int_0^A a(y)\,\frac{|\nabla_x f|^2}{f}dy\right|\left|\int_0^A \frac{f}{a(y)}dy\right| dx$$
$$\le P(\Omega)\,\frac{1+A}{a_*}\,\|M\|_{L^\infty_x}\int_0^1\int_0^\infty a(y)\frac{|\nabla_x f|^2}{f}\,dydx$$

since we have $\frac{1}{a(y)} \le \frac{1+y}{a_*} \le \frac{1+A}{a_*}$ for $y\in[0,A]$.

Estimates used in the proof: moments

$$\sup_{t \geq t_* > 0} \mathcal{M}_p(f)(t) \leq (2^{2p}\, Cst)^p$$

Cf. A. Bobylev, I. Gamba, V. Panferov.

Steps of the proof:

1. $$\mathcal{M}_p(f)(t_0) < +\infty \quad \Rightarrow \quad \sup_{t \geq t_0} \mathcal{M}_p(f)(t) \leq Cst \left(\mathcal{M}_p(f)(t_0) + (2^{2p}\, Cst)^p \right).$$

   Difficulty: track the p dependence.

2. $$\mathcal{M}_p(f)(1+t) \leq Cst\,(1 + \mathcal{M}_2(f)(t)).$$

   Difficulty: build a sequence of times between t and $t+1$ allowing to pass from $\mathcal{M}_k(f)$ to $\mathcal{M}_{k+1}(f)$.

3. $$\mathcal{M}_2(f)(t) \leq Cst\,(1 + t^{-Cst}).$$

   Difficulty: Use De la Vallée-Poussin's meth., Cf. S. Mischler, B. Wennberg.