Convex representations and their geodesic flows

joint work with

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,

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- boundary at infinity
- Gromov geodesic flow

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- limit curves
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Elevating the ending lamination conjecture?

The Bowditch "definition"

- A *non elementary hyperbolic group* Γ has a *boundary at infinity* $\partial_\infty\Gamma$ which is a *perfect metrizable compactum* (= compact metric space without isolated points) on which Γ acts as *a convergence group*: the action on

$$\partial_\infty\Gamma^{3*} = \{(x, y, z) \in \partial_\infty\Gamma^3 \mid x \neq y \neq z \neq x\},$$

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- **Example:** Surface groups.

Gromov geodesic flow

There exists a proper cocompact action of Γ on

$$\widetilde{U_0\Gamma} := \partial_\infty \Gamma^{2*} \times \mathbb{R}$$

- commuting with the $\mathbb{R}$ action,
- unique "up to reparametrisation" once one imposes natural extra conditions. *Gromov, Matheus, Champetier, Mineyev...*
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- Gromov, Coornaert–Papadopoulos developed a symbolic coding for this flow which is finite to one.

Convex representation

- A representation ρ of a hyperbolic group Γ in $\mathsf{SL}(E)$ is *convex* if there exists continuous ρ-equivariant maps ξ and θ, called *limit maps* from $\partial_\infty\Gamma$ to $\mathbb{P}(E)$ and $\mathbb{P}(E^*)$ respectively so that

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- **A construction:** the associated flat bundle over $\mathsf{U}_0\Gamma$:

$$E_\rho := \left(\widetilde{\mathsf{U}_0} \times E\right)/\Gamma.$$

decomposes, parallelly along the flow, as

$$E_\rho = \Xi \oplus \Theta,$$

with $\Xi_{(x,y,t)} := \xi(x)$ and $\Theta_{(x,y,t)} := \ker(\theta(y))$.

Convex Anosov representation

- Let M be a compact space quipped with a flow ϕ_t and Φ_t be a lift of ϕ_t on some vector bundle F.
  Then F is *contracted by the flow* is

$$\exists T_0 > 0, \quad \forall u \in F, \quad \|\Phi_{T_0}(u)\| \leqslant \frac{1}{2}\|u\|.$$

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- Convex Anosov $\rightsquigarrow$ Wanosov?

Examples

- Hitchin representations
- [Guichard–Wienhard] (G, P)-Anosov representations: there exists a representation α of G so that if ρ is (G, P)-Anosov then $\alpha \circ \rho$ is convex Anosov.
- [Guichard–Wienhard] A convex irreducible representation is convex Anosov.
- Rank 1 convex cocompact $\rightsquigarrow$ convex Anosov.
- Benoist groups: acting cocompactly on a projective strict convex set.
- Small deformations of the above.

Properties

- Every matrix $\rho(\gamma)$ is *proximal*: maximal eigenvalue $\lambda_\rho(\gamma)$ of multiplicity one / one attractive fixed point on $\mathbb{P}(E)$.
  "Anosov=proximality spreads nicely"
- The representation is *well displacing*

$$A^{-1}d(\gamma) - B \leqslant \lambda_\rho(\gamma) \leqslant Ad(\gamma) + B,$$

  where $d(\gamma) := \inf_\eta \|\eta.\gamma.\eta^{-1}\|$.
- [Delzant–Guichard–L–Mozes] ρ is a quasiisometry.
- Injective, discrete image.
- [Kapovich–Leeb–Porti] have a more algebraic characterisation.

The geodesic flow of a convex representation

- Let

  $\widetilde{\mathsf{U}_\rho\Gamma} := \{(u,v,x,y) \in E \times E^*\partial_\infty\Gamma^{2*} \mid \langle u|v\rangle = 1, u \in \xi(x), v \in \theta(y)\}$

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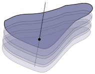
- **Theorem**
  [Geodesic flow for Convex Anosov] *The action of* Γ *on* $\widetilde{\mathsf{U}_\rho\Gamma}$ *is proper and cocompact. The corresponding flow is orbit equivalent to the Gromov geodesic flow. Moreover the flow is a metric Anosov (Smale) flow.*

Metric Anosov flow

- A *lamination* $\mathcal{F}$= a foliation for a topological space. Two laminations define a *product structure* if in any small open sets they can be described as the two factors of a product.



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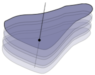
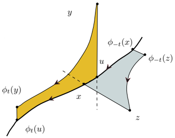


- A flow ϕ_t is *metric Anosov* if There exists two foliations $\mathcal{F}^{\pm}$ invariant by the flow, with "product structure" and $\mathcal{F}^{+}$ and $\mathcal{F}^{-}$ are contracted towards the future and past respectively.



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The result is useful and a necessary step to obtain to obtain a 1-1 coding and use the thermodynamical formalism. But:

- We do not know of hyperbolic groups admitting convex representations which are not abstractly rank 1- convex cocompact groups or Benoist groups (in which case the Anosov character of the geodesic flow is well known)
- Does there exists a hyperbolic group whose geodesic flow is not Anosov?

Elevating the ending lamination conjecture?

What can we say of representations ρ which are limits of convex Anosov representations? In particular for the Barbot component of $\mathsf{SL}(3,\mathbb{R})$?

- **Definition**: *without incidental parabolics*:= being limits+all $\rho(\gamma)$ are proximal.

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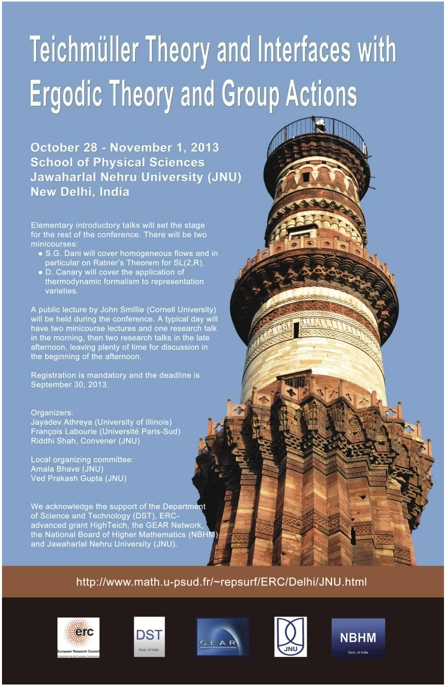
- **Question:** Existence of Cannon–Thurston maps ?
- **Question:** How to associate "ending invariants" to such representations?


Teichmüller Theory and Interfaces with Ergodic Theory and Group Actions
October 28 - November 1, 2013
School of Physical Sciences
Jawaharlal Nehru University (JNU)
New Delhi, India
Elementary introductory talks will set the stage for the rest of the conference. There will be two minicourses:
• S.G. Dani will cover homogeneous flows and in particular on Ratner's Theorem for SL(2,R),
• D. Canary will cover the application of thermodynamic formalism to representation varieties.
A public lecture by John Smillie (Cornell University) will be held during the conference. A typical day will have two minicourse lectures and one research talk in the morning, then two research talks in the late afternoon, leaving plenty of time for discussion in the beginning of the afternoon.
Registration is mandatory and the deadline is September 30, 2013.
Organizers:
Jayadev Athreya (University of Illinois)
François Labourie (Université Paris-Sud)
Riddhi Shah, Convener (JNU)
Local organizing committee:
Amala Bhave (JNU)
Ved Prakash Gupta (JNU)
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