Geometric structures on the Figure Eight Knot Complement

ICERM Workshop
Exotic Geometric Structures

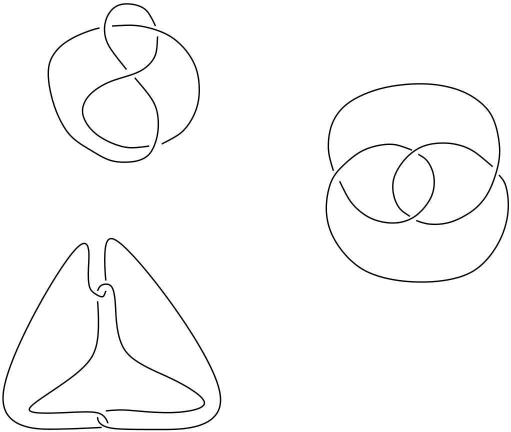
Martin Deraux

Institut Fourier - Grenoble

Sep 16, 2013

The figure eight knot

Various pictures of 4_1:



$K =$ figure eight knot

The complete (real) hyperbolic structure

$M = S^3 \setminus K$ carries a complete hyperbolic metric

M can be realized as a quotient

$$\Gamma \setminus H^3_{\mathbb{R}}$$

where $\Gamma \subset PSL_2(\mathbb{C})$ is a lattice (discrete group with quotient of finite volume)

- One cusp with cross-section a torus.
- Discovered by R. Riley (1974)
- Part of a much more general statement about knot complements/3-manifolds (Thurston)

Holonomy representation

For example by Wirtinger, get

$$\pi_1(M) = \langle g_1, g_2, g_3 \,|\, g_1 g_2 = g_2 g_3,\ g_2 = [g_3, g_1^{-1}]\,\rangle,$$

with fundamental group of the boundary torus generated by

$$g_1 \text{ and } [g_3^{-1}, g_1][g_1^{-1}, g_3]$$

Alternatively

$$\pi_1(M) = \langle a, b, t \,|\, tat^{-1} = aba,\ tbt^{-1} = ab\,\rangle.$$

The figure eight knot complement fibers over the circle, with punctured torus fiber.

Holonomy representation (cont.)

Search for $\rho : \pi_1(M) \to PSL_2(\mathbb{C})$ with $\rho(g_1) = G_1$, $\rho(g_3) = G_3$,

$$G_1 = \begin{pmatrix} 1 & 1 \\ 0 & 1 \end{pmatrix}, \quad G_3 = \begin{pmatrix} 1 & 0 \\ -a & 1 \end{pmatrix}$$

Requiring

$$G_1[G_3, G_1^{-1}] = [G_3, G_1^{-1}]G_3$$

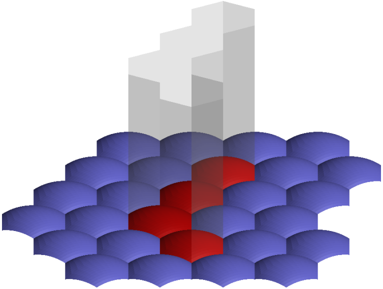
get $a^2 + a + 1 = 0$, so

$$a = \frac{-1 \pm i\sqrt{3}}{2} = \omega \text{ or } \overline{\omega}.$$

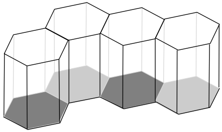
Ford domain for the image of ρ

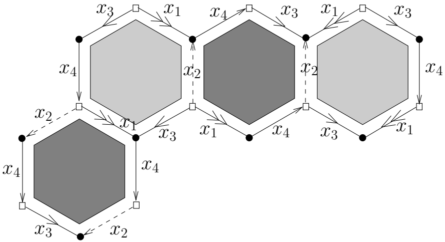
Bounded by unit spheres centered in $\mathbb{Z}[\omega]$, $\omega = \frac{-1+i\sqrt{3}}{2}$

Cusp group generated by translations by 1 and $2i\sqrt{3}$.



Prism picture





Triangulation picture

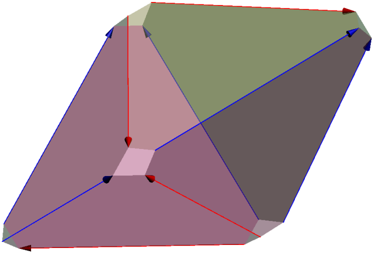
Can also get the hyperbolic structure gluing two **ideal tetrahedra**, with invariants z, w.

Compatibility equations:

$$z(z-1)w(w-1) = 1$$

For a **complete** structure, ask the boundary holonomy to have derivative 1, and this gives

$$z = w = \omega$$



Complete spherical CR structures

Spherical CR structure arising as the **boundary of a ball quotient**.

Ball quotient: $\Gamma \setminus \mathbb{B}^2$, where Γ is a discrete subgroup of $\mathrm{Bihol}(\mathbb{B}^2) = PU(2,1)$.

The manifold at infinity inherits a natural spherical CR structure, called "complete" or "uniformizable".

Domain of discontinuity

$\Gamma \subset PU(2,1)$ discrete

- Domain of discontinuity Ω_Γ
- Limit set $\Lambda_\Gamma = S^3 - \Omega_\Gamma$

The orbifold/manifold at infinity of Γ is $\Gamma \setminus \Omega_\Gamma$

- Manifold only if no fixed points in Ω_Γ (isolated fixed points inside $\mathbb{B}^2$ are OK);
- Can be empty (e.g. when Γ is (non-elementary) and a normal subgroup in a lattice).

Biholomorphisms of $\mathbb{B}^2$

Up to scaling, $\mathbb{B}^2$ carries a unique metric invariant under the $PU(2,1)$-action, the Bergman metric.

$$\mathbb{B}^2 \subset \mathbb{C}^2 \subset P^2_{\mathbb{C}}$$

With this metric: **complex hyperbolic plane**.

- Biholomorphisms of $\mathbb{B}^2$: restrictions of projective transformations (i.e. linear tsf of $\mathbb{C}^3$).
- $A \in GL_3(\mathbb{C})$ preserves $\mathbb{B}^2$ if and only if

$$A^*HA = H$$

where

$$H = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$$

Equivalent Hermitian form:

$$J = \begin{pmatrix} 0 & 0 & 1 \\ 0 & 1 & 0 \\ 1 & 0 & 0 \end{pmatrix}$$

Siegel half space:

$$2\mathfrak{Im}(w_1) + |w_2|^2 < 0$$

Boundary at infinity $\partial_\infty H^2\mathbb{C}$ (minus a point) should be viewed as the **Heisenberg group**, $\mathbb{C} \times \mathbb{R}$ with group law

$$(z, t) * (w, s) = (z + w, t + s + 2\mathfrak{Im}(z\bar{w})).$$

- Copies of $H^1_{\mathbb{C}}$ (affine planes in $\mathbb{C}^2$) have curvature -1
- Copies of $H^2_{\mathbb{R}}$ ($\mathbb{R}^2 \subset \mathbb{C}^2$) have curvature $-1/4$ (linear only when through the origin)
- No totally geodesic embedding of $H^3_{\mathbb{R}}$!

For this normalization, we have

$$\cosh \frac{1}{2} d(z,w) = \frac{|\langle Z, W \rangle|}{\sqrt{\langle Z, Z \rangle \langle W, W \rangle}}$$

where

- Z are homogeneous coordinates for z
- W are homogeneous coordinates for w

Isometries of $H^2_{\mathbb{C}}$

Classification of (non-trivial) isometries

- **Elliptic** ($\exists$ fixed point inside)
  - regular elliptic (three distinct eigenvalues)
  - complex reflections
    - in lines
    - in points
- **Parabolic** (precisely one fixed point in $\partial H^2_{\mathbb{C}}$)
  - Unipotent (some representative has 1 as its only eigenvalue)
  - Screw parabolic
- **Loxodromic** (precisely two fixed points in $\partial H^2_{\mathbb{C}}$)

$PU(2,1)$ has index 2 in $\mathrm{Isom} H^2_{\mathbb{C}}$ (complex conjugation).

Central question

Which 3-manifolds admit a complete spherical CR structure?

In other words:
Which 3-manifolds occur as the manifold at infinity $\Gamma \setminus \Omega_\Gamma$ of some discrete subgroup $\Gamma \subset PU(2,1)$?

Silly example: lens spaces! Take Γ generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/q} & 0 \\ 0 & 0 & e^{2\pi ip/q} \end{pmatrix}$$

with p, q relatively prime integers (in this case $\Omega_\Gamma = S^3$).

Central question

Which 3-manifolds admit a complete spherical CR structure?

In other words:
Which 3-manifolds occur as the manifold at infinity $\Gamma \setminus \Omega_\Gamma$ of some discrete subgroup $\Gamma \subset PU(2,1)$?

Silly example: lens spaces! Take Γ generated by

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & e^{2\pi i/q} & 0 \\ 0 & 0 & e^{2\pi i p/q} \end{pmatrix}$$

with p, q relatively prime integers (in this case $\Omega_\Gamma = S^3$).

More complicated examples

- Nil manifolds
- Lots of circle bundles (Anan'in-Gusevski, Falbel, Parker,......)

(open hyperbolic manifolds)

- Whitehead link complement (Schwartz, 2001)
- Figure eight knot complement (D-Falbel, 2013)
- Whitehead link complement (Parker-Will, 201k, $k \geq 3$)

(closed hyperbolic manifolds)

- ∞ many closed hyperbolic manifolds (Schwartz, 2007)
- ∞ many surgeries of the figure eight knot (D, 201k, $k \geq 3$).

More complicated examples

- Nil manifolds
- Lots of circle bundles (Anan'in-Gusevski, Falbel, Parker,......)

(open hyperbolic manifolds)

- Whitehead link complement (Schwartz, 2001)
- Figure eight knot complement (D-Falbel, 2013)
- Whitehead link complement (Parker-Will, 201k, $k \geq 3$)

(closed hyperbolic manifolds)

- ∞ many closed hyperbolic manifolds (Schwartz, 2007)
- ∞ many surgeries of the figure eight knot (D, 201k, $k \geq 3$).

More complicated examples

- Nil manifolds
- Lots of circle bundles (Anan'in-Gusevski, Falbel, Parker,. . . . . .)

(open hyperbolic manifolds)

- Whitehead link complement (Schwartz, 2001)
- Figure eight knot complement (D-Falbel, 2013)
- Whitehead link complement (Parker-Will, 201k, $k \geq 3$)

(closed hyperbolic manifolds)

- ∞ many closed hyperbolic manifolds (Schwartz, 2007)
- ∞ many surgeries of the figure eight knot (D, 201k, $k \geq 3$).

Negative result

(not all manifolds)

- Goldman (1983) classifies T^2-bundles with spherical CR structures.

For instance T^3 admits no spherical CR structure at all (complete or not!)

CR surgery (Schwartz 2007)

M a complete spherical CR structure on an open manifold with torus boundary.

If we have

1. The holonomy representation deforms
2. $\Gamma \setminus \Omega$ is the union of a compact region and a "horotube"
3. Limit set is porous

Then ∞ many Dehn fillings $M_{p/q}$ admit a complete spherical CR structures.

Not effective, which p/q work?

The figure eight knot

Theorem

(D-Falbel, 2013) The figure eight knot complement admits a complete spherical CR structure.

Relies heavily on:

Theorem

(Falbel 2008) Up to $\widehat{PU}(2,1)$-conjugacy, there are precisely three boundary unipotent representations $\rho_1, \rho_2, \rho_3 : \pi_1(M) \to PU(2,1)$.

The figure eight knot

Theorem

(D-Falbel, 2013) The figure eight knot complement admits a complete spherical CR structure.

Relies heavily on:

Theorem

(Falbel 2008) Up to $\widehat{PU}(2,1)$-conjugacy, there are precisely three boundary unipotent representations $\rho_1, \rho_2, \rho_3 : \pi_1(M) \to PU(2,1)$.

Preliminary analysis

Falbel 2008:

- $Im(\rho_1) \lhd PU(2,1,\mathbb{Z}[\omega])$
- $Im(\rho_2) \subset PU(2,1,\mathbb{Z}[\sqrt{-7}])$

In particular, ρ_1 and ρ_2 are discrete.

$\mathrm{Im}(\rho_1)$ has empty domain of discontinuity (same limit set as the lattice $PU(2,1,\mathbb{Z}[\omega])$).

Action of $\mathrm{Out}(\pi_1(M)) \rightsquigarrow$ up to conjugation,

$$\rho_3 = \rho_2 \circ \tau$$

for some outer automorphism τ of $\pi_1(M)$ (orientation reversing homeo of M).

Preliminary analysis

Falbel 2008:

- $Im(\rho_1) \lhd PU(2,1,\mathbb{Z}[\omega])$
- $Im(\rho_2) \subset PU(2,1,\mathbb{Z}[\sqrt{-7}])$

In particular, ρ_1 and ρ_2 are discrete.

$\mathrm{Im}(\rho_1)$ has empty domain of discontinuity (same limit set as the lattice $PU(2,1,\mathbb{Z}[\omega])$).

Action of $\mathrm{Out}(\pi_1(M)) \rightsquigarrow$ up to conjugation,

$$\rho_3 = \rho_2 \circ \tau$$

for some outer automorphism τ of $\pi_1(M)$ (orientation reversing homeo of M).

Complete structure for 4_1.

Write

- $\Gamma = \mathrm{Im}(\rho_2)$,
- $G_k = \rho_2(g_k)$.

$$G_1 = \begin{pmatrix} 1 & 1 & -\frac{1+\sqrt{(7)}i}{2} \\ 0 & 1 & -1 \\ 0 & 0 & 1 \end{pmatrix} \quad G_2 = \begin{pmatrix} 2 & \frac{3-i\sqrt{7}}{2} & -1 \\ -\frac{3+i\sqrt{7}}{2} & -1 & 0 \\ -1 & 0 & 0 \end{pmatrix}$$

$$G_3 = G_2^{-1} G_1 G_2.$$

- G_1, G_3 unipotent
- G_2 regular elliptic of order 4.

Back to Dirichlet domains

To see that ρ_2 does the job, one way is to study **Dirichlet/Ford domains** for $\Gamma_2 = \mathrm{Im}\rho_2$.

$$D_{\Gamma,p_0} = \{z \in \mathbb{B}^2 \,:\, d(z,p_0) \leq d(\gamma z, p_0)\ \forall \gamma \in \Gamma\}$$

We will assume D_{Γ,p_0} has non empty interior (hard to prove!)

Key:

1. When no nontrivial element of Γ fixes p_0, this gives a *fundamental domain* for the action of Γ.
2. Otherwise, get a fundamental domain for a coset decomposition (cosets of $\mathrm{Stab}_\Gamma p_0$ in Γ).

More subtle:

1. Beware these often have infinitely many faces (Phillips, Goldman-Parker)
2. Depend heavily on the *center* p_0.

Back to Dirichlet domains

To see that ρ_2 does the job, one way is to study **Dirichlet/Ford domains** for $\Gamma_2 = \mathrm{Im}\rho_2$.

$$D_{\Gamma,p_0} = \{z \in \mathbb{B}^2 \,:\, d(z,p_0) \leq d(\gamma z, p_0)\ \forall \gamma \in \Gamma\}$$

We will assume D_{Γ,p_0} has non empty interior (hard to prove!)

Key:

1. When no nontrivial element of Γ fixes p_0, this gives a *fundamental domain* for the action of Γ.
2. Otherwise, get a fundamental domain for a coset decomposition (cosets of $\mathrm{Stab}_\Gamma p_0$ in Γ).

More subtle:

1. Beware these often have infinitely many faces (Phillips, Goldman-Parker)
2. Depend heavily on the *center* p_0.

Back to Dirichlet domains

To see that ρ_2 does the job, one way is to study **Dirichlet/Ford domains** for $\Gamma_2 = \mathrm{Im}\rho_2$.

$$D_{\Gamma,p_0} = \left\{z \in \mathbb{B}^2 \,:\, d(z,p_0) \leq d(\gamma z, p_0)\ \forall \gamma \in \Gamma\right\}$$

We will assume D_{Γ,p_0} has non empty interior (hard to prove!)

Key:

1. When no nontrivial element of Γ fixes p_0, this gives a *fundamental domain* for the action of Γ.
2. Otherwise, get a fundamental domain for a coset decomposition (cosets of $\mathrm{Stab}_\Gamma p_0$ in Γ).

More subtle:

1. Beware these often have infinitely many faces (Phillips, Goldman-Parker)
2. Depend heavily on the *center* p_0.

Poincaré polyhedron theorem

Important tool for proving that D_{Γ,p_0} has non-empty interior.

Use D_{F,p_0} instead of D_{Γ,p_0} for some subset $F \subset \Gamma$.
[In simplest situations, F is *finite*].

Assume

- F generates Γ;
- $F = F^{-1}$ and opposite faces are isometric;
- Cycle conditions on ridges (faces of codimension 2)
- Cycles of infinite vertices are parabolic

Then the group Γ is **discrete** and we get

- An explicit group presentation $\langle F|R\rangle$ where R are cycle relations.
- A list of orbits of fixed points.

Dirichlet/Ford domains

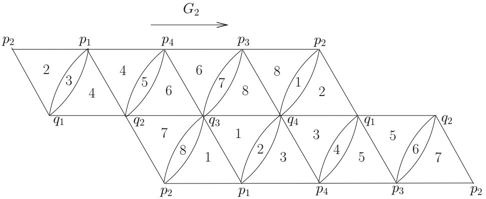
Two natural choices for the center:

- Fixed point of G_1 (unipotent) $\rightsquigarrow$ Ford domain
- Fixed point of G_2 (regular elliptic) $\rightsquigarrow$ Dirichlet domain

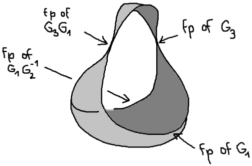
Dirichlet: $\partial_\infty D_\Gamma$ is (topologically!) a **solid torus**.

Ford: $\partial_\infty D_\Gamma$ is a **pinched solid torus** (horotube).

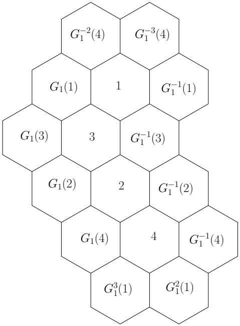
Combinatorial structure of $\partial(\partial_\infty D_\Gamma)$ for the Dirichlet domain domain



$$1 \quad : \quad G_1 p_0$$
$$2 \quad : \quad G_1^{-1} p_0$$

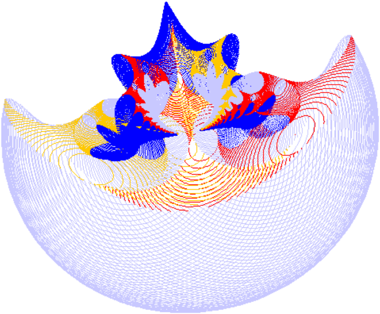


Combinatorial structure of $\partial(\partial_\infty D_\Gamma)$ for the Ford domain



$1 : G_2 p_0 \quad 2 : G_2^{-1} p_0$

$3 : G_3 p_0 \quad 4 : G_3^{-1} p_0$



Computational techniques

The key is to be able to **prove** the combinatorics of Dirichlet/Ford polyhedra.

Need to solve a system of quadratic inequalities in four variables.

- Guess faces and incidence by numerical computations (grids in parameters for intersections of two bisectors)
- To prove your guess:
  - Exact computation in appropriate number field, or
  - Use genericity

Γ is a triangle group!

The Poincaré polyhedron theorem gives the following presentation:

$$\langle\, G_1, G_2 \mid G_2^4, (G_1 G_2)^3, (G_2 G_1 G_2)^3 \,\rangle$$

The group has index 2 in a $(3,3,4)$-triangle group.

Deformations

Using real $(3, 3, 4)$-triangle groups, one gets a 1-parameter family of deformations, where the unipotent element becomes elliptic with eigenvalues $(1, \zeta, \overline{\zeta})$, $|\zeta| = 1$.

These deformations give complete spherical CR structures on $(k, 1)$-surgeries of the figure eight knot (with $k \geq 5$).

FKR Census

		Zero-dimensional prime components						
		Number(s) of Solutions						
Name	1-D	Ext. Degrees	$PGL_3(\mathbb{C})$	$PSL_3(\mathbb{R})$	$PSL_2(\mathbb{C})$	$PSL_2(\mathbb{R})$	$PU(2,1)$	*Volumes*
m003	2	2, 2, 8, 8	20	0	2	0	2	0.648847 **2.029883**
m004	0	2, 2, 2, 2	8	0	2	0	6	**2.029883**
m006	2	6, 6, 12, 28	43	1	3	1	15	0.707031 0.719829 0.971648 1.284485 **2.568971**
m007	0	3, 6, 8, 8, 8	33	1	3	1	15	0.707031 0.822744 1.336688 **2.568971**
m009	0	2, 4, 4, 4, 6, 8	28	2	2	0	8	0.507471 0.791583 1.417971 **2.666745**
m010	2	4, 6, 6, 12, 12	38	0	2	0	4	0.251617 0.791583 0.809805 0.982389 1.323430 **2.666745**
m011	1	3, 4, 16, 64	87	5	7	3	21	0.226838 0.251809 0.328272 0.397457 0.452710 0.643302 0.685598 0.700395 0.724553 0.770297 0.879768 **0.942707** 0.988006 1.099133 1.184650 1.846570 **2.781834**
m015	0	3, 4, 4, 6, 6	23	3	3	1	11	0.794323 1.583167 **2.828122**
m016	1	3, 3, 10, 50	66	4	6	4	24	0.296355 0.403707 0.710033 0.753403 0.773505 0.796590 0.886451 1.135560 1.422985 1.505989 **2.828122**
m017	3	3,4,6,6,44	63	1	3	1	21	0.527032 0.794323 0.801984 0.828705 1.252969 1.588647 **2.828122**
m019	1	4, 4, 22, 84	114	6	8	4	24	0.027351 0.062112 0.323395 0.332856 0.347159 0.411244 0.467624 0.524801 0.544151 0.599455 0.638404 0.738805 0.758111 0.798098 0.851139 0.916588 1.101800 1.130263 1.190919 **1.263709** 1.340255 2.111776 **2.944106**
Wh. link	0	2, 2, 4, 4, 10, 10	32	0	2	0	14	1.132196 1.683102 **3.663862**

TABLE 1. Description of the solutions

Other examples

Hyperbolic manifolds with low complexity (up to 3 tetrahedra), and representations into $PU(2,1)$ with

Unipotent and **Rank 1**,

boundary holonomy.
List from Falbel, Koseleff and Rouiller (2013):

$$\begin{aligned} &\texttt{m004} = 4_1 \text{ knot} \\ &\texttt{m009} \\ &\texttt{m015} = 5_2 \text{ knot} \end{aligned}$$

Same seems to work:

- Dirichlet domains with finitely many faces (for well-chosen center)
- The image groups are triangle groups.