Projective Structures on Manifolds

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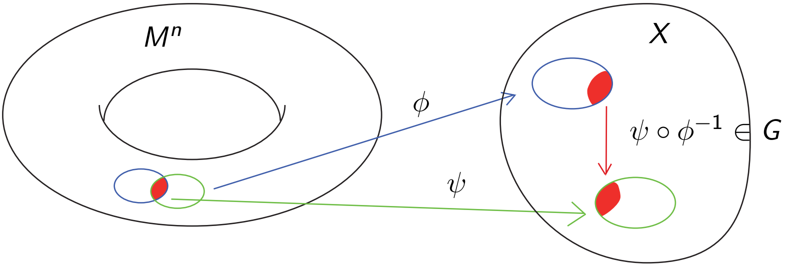
(2012) arXiv 1109.0585

joint: Darren Long, Stephan Tillmann

(G, X) = geometric structure



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Projective geometry $= (\mathrm{PGL}(n+1, \mathbb{R}), \mathbb{R}P^n)$

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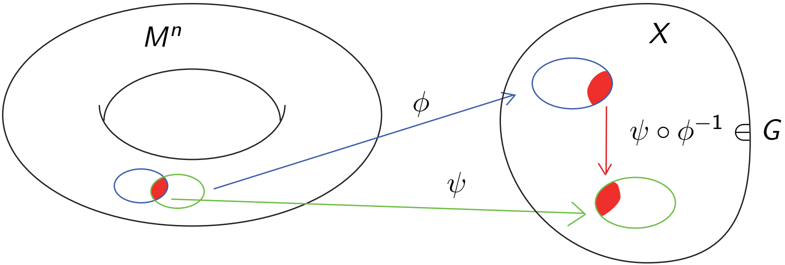

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Constant curvature geometries are "subgeometries" of projective geometry

$\therefore$ Every constant curvature n-manifold has underlying projective structure.

$\Rightarrow$ Every surface has projective structure.

M^n closed, $\pi_1 M^n = 1$, M real projective $\Rightarrow$ $M = S^n$

projective representation of (G,X) geometry

$(\rho, \text{dev}) : (G, X) \longrightarrow (\text{PGL}(n+1, \mathbb{R}), \mathbb{R}P^n) \qquad \text{dev}(g \cdot x) = (\rho g)(\text{dev}\, x)$

Molnar (1990), Thiel (1994)

8 Thurston geometries (virtually) have projective representations

$\Rightarrow$ 3-manifold with Thurston geometry (virtually) projective.

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Question are there other connected 3-manifolds with no projective structure ?

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M. Kapovich (2007) $\exists$ non-hyperbolic projective projective W^4
with sec. curv. $-1 \leq K \leq -1 + \epsilon$

Projective Surfaces are classified

Goldman (1990)
convex projective structures $\mathcal{T}_{conv\mathbb{R}P}(F)$ is cell dimension $8 \cdot |\chi(F)|$
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Labourie (∼2007), Loftin (∼2006)
Convex projective structure ↔ (Conf. str + holo cubic diff.)
$\therefore \mathcal{T}_{conv\mathbb{R}P}(F)$ is vector bundle over $\mathcal{T}(F)$

Fock + Goncharov (∼2007)
Nice coordinates for (br cover of) $\mathcal{T}_{conv\mathbb{R}P}(F)$

Affine patch $= \mathbb{R}P^n \setminus \mathbb{R}P^{n-1} \cong \mathbb{R}^n$

$\Omega \subset \mathbb{R}P^n$ properly convex if interior of compact convex in affine patch

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(properly/strictly) convex projective orbifold $Q = \Omega/\Gamma$

$PGL(n+1, \mathbb{R}) \supset Aut(\Omega) =$ subgroup preserving Ω.
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Hyperbolic n-space: $\Omega = D =$ unit ball $\quad Aut(D) = PO(n,1) \cong Isom(\mathbb{H}^n)$

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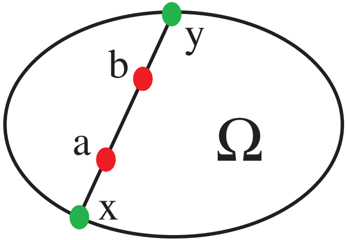
Strictly convex is generalization of hyperbolic manifold
Properly convex much more general: like curvature ≤ 0.

Benoist (2005) If M^n closed properly convex then
M strictly convex $\Leftrightarrow \pi_1 M$ is δ-hyperbolic.

Hilbert metric on Ω $\qquad d_\Omega(a,b) = |\log CR(x,a,b,y)|$



- Finsler metric.
- $\Omega = D \qquad d_\Omega = 2\cdot d_{\mathbb{H}^n}$
- $\Omega =$ interior of triangle $d_\Omega = HEX$
- Complete
- preserved by $Aut(\Omega)$
- $\Rightarrow$ volume form

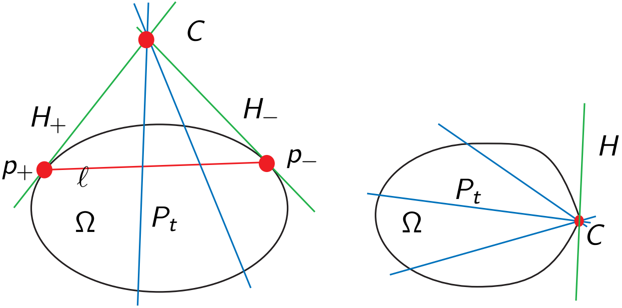
Strictly convex $\Rightarrow$ $Aut(\Omega) = Isom(d_\Omega)$ $\qquad$ (false for $\Omega =$open simplex)

de la Harpe (1991) HEX is norm on $\mathbb{C}$. unit ball is regular hexagon ⬡
$[e^{x_0} : e^{x_1} : e^{x_2}] \mapsto x_0 + x_1\omega + x_2\omega^2 \qquad \omega = e^{2\pi i/3}$

biblical metric: unique Finsler metric with $\pi = 3$

Isometries of properly convex Ω are hyperbolic/elliptic/parabolic

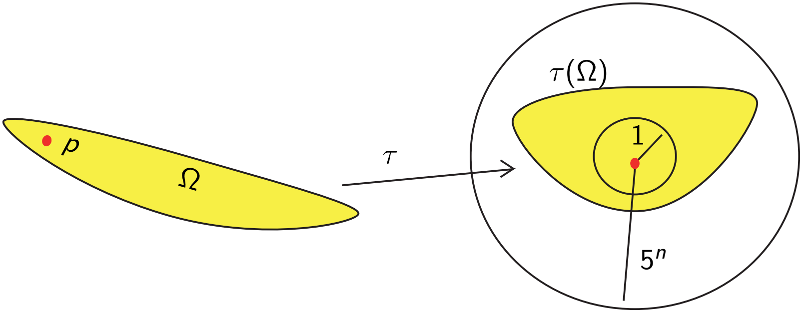
permute pencil of hyperplanes



Hyperbolic Parabolic

Question Is the set of points moved less than d by isometry connected ? convex ??

Benzécri compactness (1960) $\forall\ p \in \Omega \subset \mathbb{R}P^n$ properly convex.
$\exists\ \tau \in PGL(n+1, \mathbb{R})$ st. $\tau(p) = 0$ and $B(1) \subset \Omega \subset B(5^n)$.



$\therefore$ Hilbert geometries locally uniformly bilipschitz to Euclidean.

$V \cong \mathbb{R}^n$
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$S^1_\infty \leftrightarrow$ qfs rank 1   Mobius band outside $S^1_\infty \leftrightarrow$ qf signature (1,1)

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$n = 3$ $\dim \Omega_3 = 5$ $\partial\Omega_3 \leftrightarrow$ positive semi-definite qfs. $\supset \infty\cdot$(copies Ω_2)

$\therefore$ Many JNF in $GL(\Omega_n)$

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Auslander and Swan (1967)
every polycyclic group $\subset GL(n, \mathbb{Z})$ some n
G f.g. nilpotent $\Rightarrow$ polycyclic.
$\therefore \exists\ G \subset PGL(Pos)$ discrete group parabolics exotic cusps

But maximal cusps (vol $< \infty$) are Bieberbach groups !

Deformations of convex projective manifolds

$PC(M) \subset Hom(\pi_1 M, \mathrm{PGL}(n+1, \mathbb{R}))$
holonomies of properly convex structures on M^n
$SC(M) \cdots$ strictly convex $\cdots$

($\exists$ codimension-1 flat embedded projective submanifold in M) $\Rightarrow$ M deforms.

Koszul (1965) M closed $\Rightarrow PC(M)$ open
(C, Long, Tillmann (WIP) extend to finite volume properly convex case + end condition)

M closed $\Rightarrow SC(M)$ is closed
Choi + Goldman (1993 $n = 2$), I. Kim (2005 $n = 3$), Benoist (2005 all n)

$\exists$ Finite dimensional moduli space of deformations of closed hyperbolic M^3 often locally rigid but:

C, Long, Thistlethwaite (2006, 2007)

Of first 4500 closed hyperbolic 3-manifolds in census
61 infinitesimally deform: $H^1 \neq 0$ Why ??
Of these it is proved: 25 deform and 3 are rigid
Some that deform (e.g. vol_3) are non-Haken.
For vol_3 get free action on building for PGL_4
(cf: Culler-Shalen action on trees)

Under smoothness assumption on rep. variety:
$(\exists\ \mathbb{R}P^n \text{ deformations of } M^n_{\mathbb{R}}) \Leftrightarrow (\exists\ \mathbb{CH}^n \text{ deformations of } M_{\mathbb{C}})$

Cusp $C = \Omega/\Gamma$
Γ = group parabolics fixes $p \in \partial\Omega$ and H = supp. hyperplane at p
- preserves algebraic horospheres

$\Rightarrow C \cong (\partial C) \times [0, \infty)$
Maximal cusp if ∂C compact

$\exists$ Margulis constant μ_n for properly convex manifolds virt. nilpotent

Thickish-thinnish M^n strictly convex $\Rightarrow$ $M = A \cup B$ $A \cap B = \partial A = \partial B$
- If $\text{inj}(x) \leq 3^{-(n+1)}\mu_n$ then $x \in A$
- If $x \in A$, then $\text{inj}(x) \leq \mu_n/2$
- Each component of A is a Margulis tube or a cusp

M^n strictly convex and $vol(M) < \infty$ then every end is maximal cusp

Every maximal cusp in properly convex is hyperbolic
$= \Omega/\Gamma$ with $\Gamma \subset O(n, 1)$ fixes $p \in \partial\Omega$

properly convex $Q = SL(n, \mathbb{Z}) \backslash SL(n, \mathbb{R})/SO(n)$ and $vol(Q) < \infty$
but end not a cusp

"strictly convex = properly convex + rel. δ-hyperbolic + parabolic ends"

properly convex $M = \Omega/\Gamma$ & $\text{vol}(M) < \infty$
& M = interior compact N & holonomy each ∂N is parabolic. TFAE:

- M is strictly convex
- $\partial\Omega$ is C^1
- $\pi_1 N$ is δ-hyperbolic relative to $\cup \pi_1 \partial N$

Benoist: compact case.

Topological Finiteness

- strictly convex $dim \neq 3 \Rightarrow$ fin. many. top. types with $vol < V$
- properly convex fin. many top. types with $diam < D$ and $\exists p \ \text{inj}(p) > \epsilon$

"Volume bounds diameter when $n \geq 4$" :

strictly convex $\quad dim = n \geq 4 \quad \exists C_n \ \ \forall M^n$ closed

$$\text{diam}(M) \leq 9 \cdot \text{diam}(thick(M)) \leq C_n \cdot \text{vol}(M)$$

true for hyperbolic manifolds.

Theory of geometric transitions + Danciger, Wienhard

Continuous family of projective structures M_t on M

- (G, X)-structure for $t > 0$
- (G', X')-structure at $t = 0$

$(G, X) \rightsquigarrow (G', X')$

e.g $\mathbb{H}^n \rightsquigarrow \mathbb{E}^n$ and $\mathbb{S}^n \rightsquigarrow \mathbb{E}^n$

but $\mathbb{H}^3 \not\rightsquigarrow \mathbb{H}^2 \times \mathbb{R}$