TRANSITIONAL GEOMETRY

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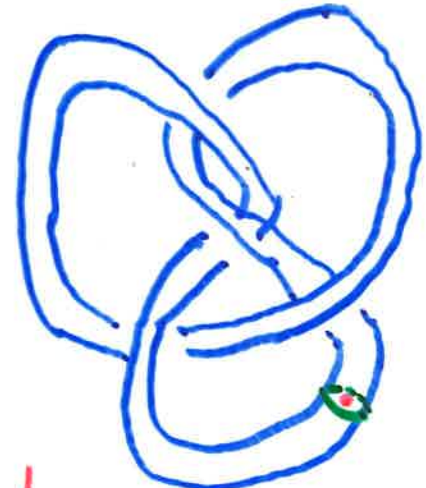
→ How are geometric structures inter-related?

→ Deform hyperbolic structures to other structures within projective geometry.

Ex: Figure 8 Knot K $\quad M = S^3 - K$



unique complete hyperbolic structure

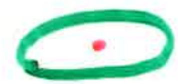
Allow cone singularities $M(\alpha)$

Increase $\boxed{\alpha \to 2\pi/3}$ $\quad \alpha \in (0, 2\pi/3)$

Volume $\to 0$ $\quad$ Diameter $\to 0$

Rescale: Curvature $\to 0$

$M(\alpha) \to$ Euclidean cone mfld E

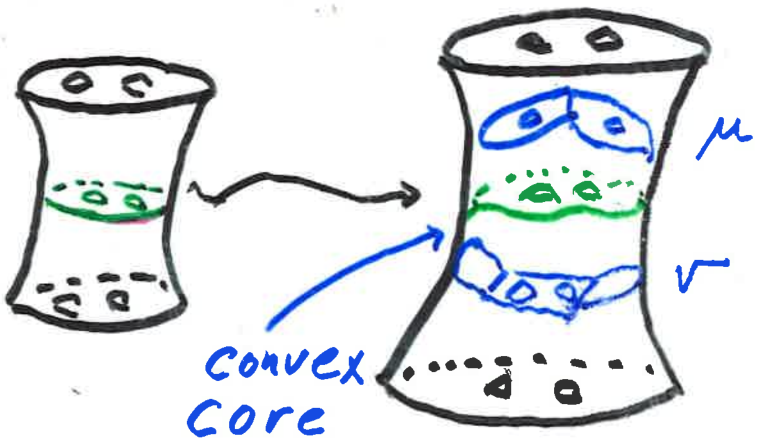


Also $M(\alpha)$ Spherical, $\boxed{\alpha \in (2\pi/3, \pi]}$

$\underset{\text{rescaled}}{M(\alpha) \longrightarrow E}$ $\quad \alpha \to 2\pi/3$

Quasi-Fuchsian Transition

Fuchsian quasi-Fuchsian



$(t\mu, t\nu) \xrightarrow[t\to 0]{} m \in$ Fuchsian

$m =$ unique min of

$$l_\mu + l_\nu : T_g \to \mathbb{R}$$

(Bonahon, Series)

(μ, ν) = bending lamination

$\mathcal{FML}_+$: ① (μ, ν) s.t. $\mu \cup \nu$ fill up

② bending $< \pi$ along $\mathcal{S}$

$\forall (\mu,\nu) \in \mathcal{FML}_+$ $\exists$ quasi-Fuchsian bent along (μ,ν) (Bonahon–Otal)

unique if $\in \mathcal{S} \times \mathcal{S}$

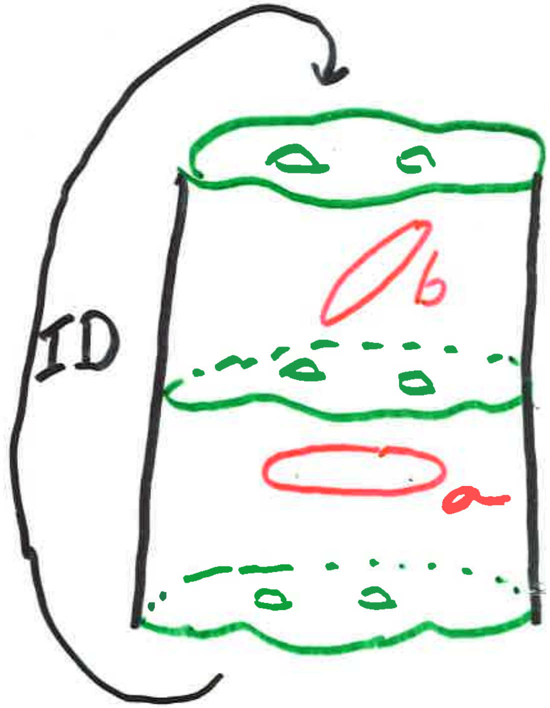
(local rigidity: Hodgson –)

unique near Fuchsian

(Bonahon, Series)

Assume $(\mu,\nu) = (a,b) \in \mathcal{S} \times \mathcal{S}$

Double along convex boundary



$S \times S^1$ w/ cone singularities along a, b w/ angles

$\alpha, \beta = 2(\pi - \theta_a), 2(\pi - \theta_b)$

$\alpha, \beta \longrightarrow 2\pi$ as $t \to 0$ $\theta_a = t = \theta_b$

Collapse to $\mathbb{H}^2$

Transition ?

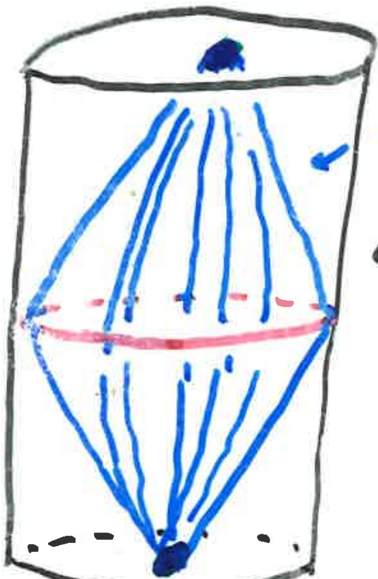
To what ?

Analog of E^3: Half-pipe geometry (HP)

(Danciger)

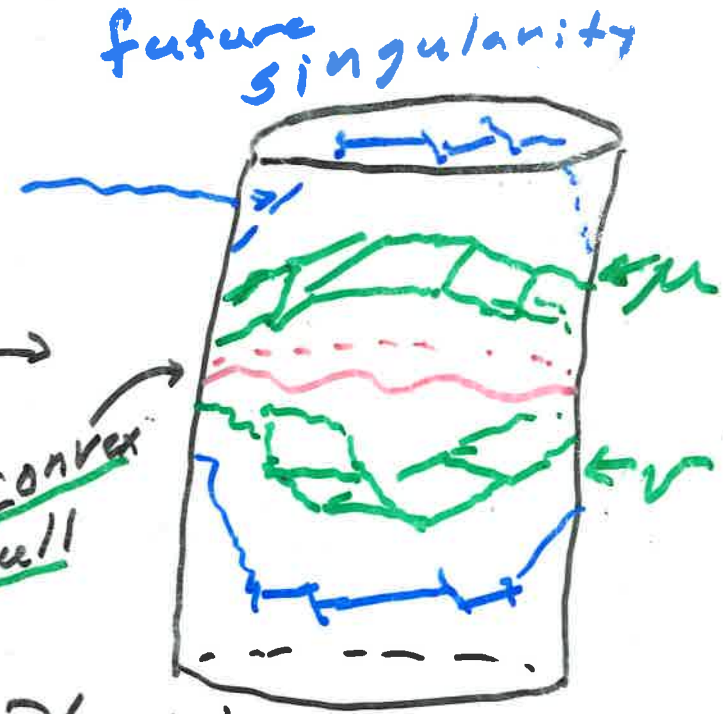
Analog of S^3: Anti-de Sitter geometry (AdS)

(Mess) Globally Hyperbolic doubled



$\Omega =$ Domain of discontinuity

deform

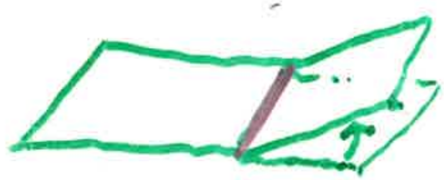


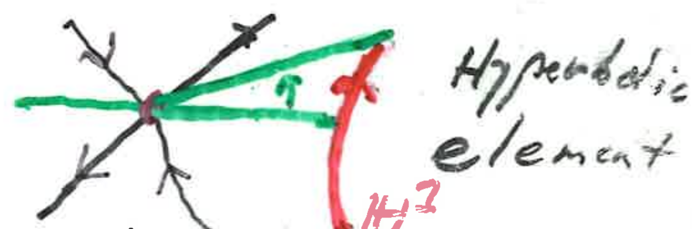
$C =$ convex hull

Fuchsian

$\Omega/\Gamma_F \approx \Sigma \times \mathbb{R}$

$\partial(C/\Gamma)$ is "bent" along μ, ν





"Bending" is hyperbolic "boost"

Geometry of Collapse to $\mathbb{H}^2$

$M = \Sigma \times S^1 - (\alpha \cup \beta)$ $\qquad \pi = \pi_1 M$

Have $\rho_s : \pi \to SO(3,1)$ $\qquad s > 0$

$\qquad \rho_0 \longrightarrow SO(2,1)$ $\qquad s = 0$

$\qquad \rho_s \longrightarrow SO(2,2)$ $\qquad s < 0$

Quadratic forms $g_s = -x_1^2 + x_2^2 + x_3^2 + s x_4^2$

$G_s \subset PGL(4,\mathbb{R})$ preserving g_s

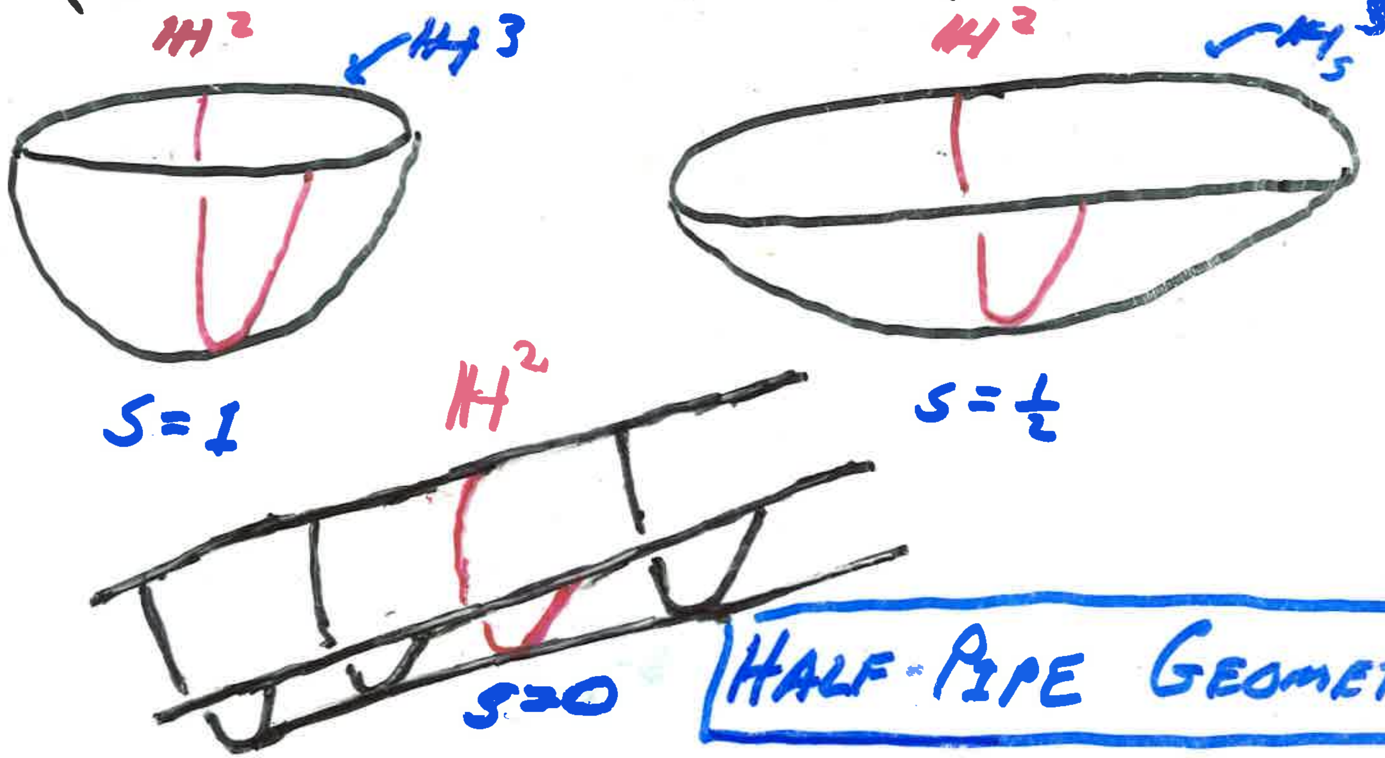
$s > 0 \leftrightarrow \mathbb{H}^3$ $\qquad s < 0 \leftrightarrow AdS^3$

$\mathbb{H}^2 \leftrightarrow x_4 = 0$

$s = 0$ $\quad \underbrace{\left(\begin{array}{c|c} SO(2,1) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline 0\ 0\ 0 & 1 \end{array}\right)}_{G_0}$ $\quad \rho_0 \overset{\text{lift}}{\rightsquigarrow} \text{to } \boxed{\tilde{\rho}_0}$ $\quad \underbrace{\left(\begin{array}{c|c} SO(2,1) & \begin{matrix}0\\0\\0\end{matrix} \\ \hline x\ y\ z & 1 \end{array}\right)}_{\tilde{G}_0}$ $\quad \begin{pmatrix} x \\ y \\ z \end{pmatrix} \in \mathbb{R}^{2,1}$

Set $\{X \in \mathbb{R}^4 \mid g_s(X) = -1\}$

(Picture is reduced by 1 dimension)

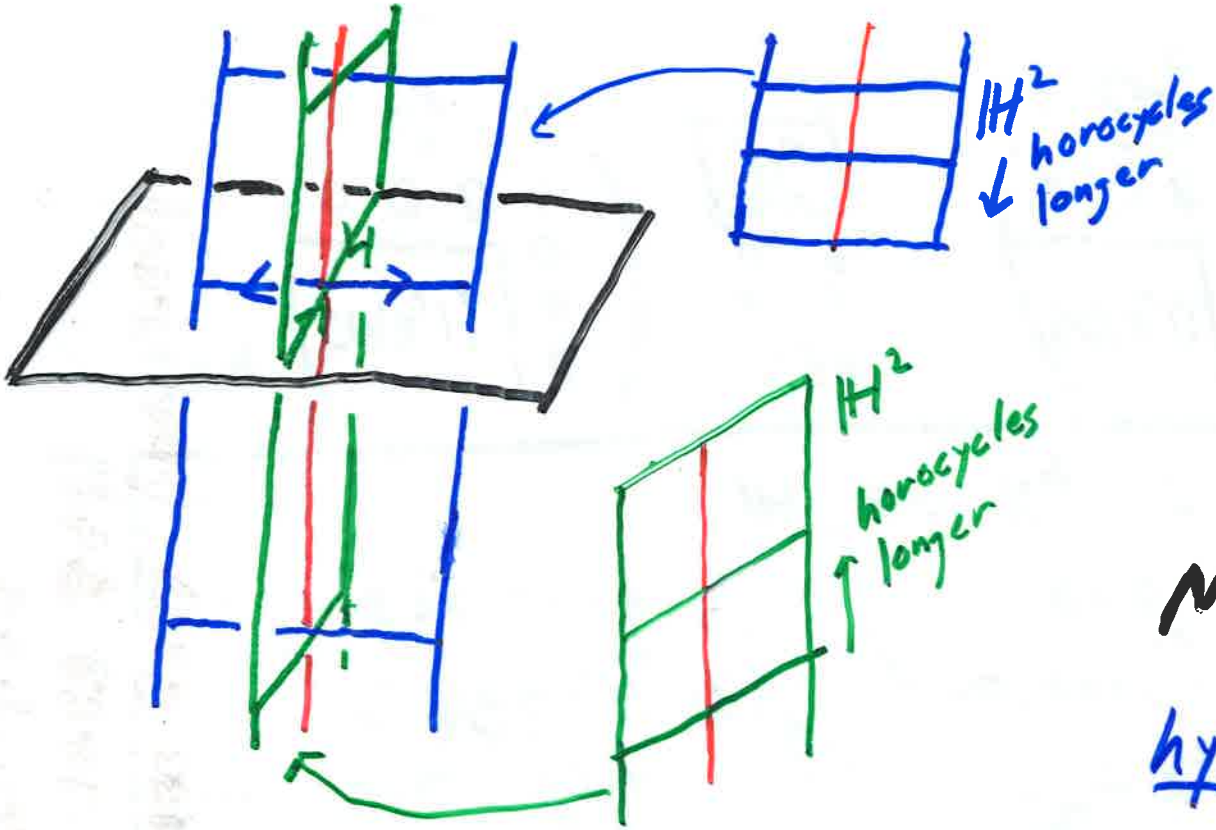


HALF-PIPE GEOMETRY

$(\tilde{G}_0, X)$ - structure $X \approx \mathbb{H}^2 \times \mathbb{R}$

preserves $g_0 = -x_1^2 + x_2^2 + x_3^2$

Sol Geometry Transition



$T^2 \times I / (x,0) \sim (\varphi(x), 1)$

$\varphi : T^2 \circlearrowleft$ Anosov

Sol geometry

$M_\varphi = (T^2 - pt) \times I / \sim_\varphi$

hyperbolic geometry

complete structure + w/ cone angles $\alpha < 2\pi$

$\alpha \to 2\pi$ collapse to $\mathbb{H}^2$ (either one)

(Heusner-Porti-Suarez)

(HPS) Can regenerate Sol to $\mathbb{H}^3$

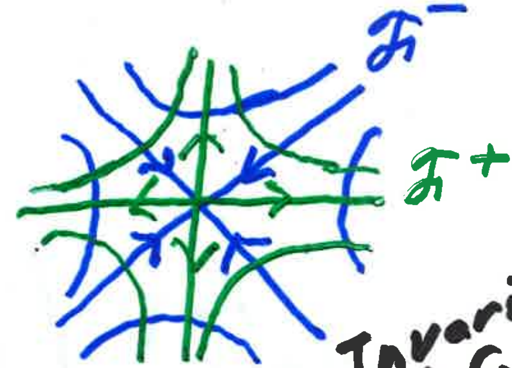
(Danciger) There is a transition to (singular) AdS

Higher genus? φ pseudo-Anosov

M^3_φ has singular Sol structure

w/ cone angles $k\pi$ $\quad k \geq 3$

Also has hyperbolic structure (Thurston)



Invariant measured foliations

Transition?

$\underline{\mathrm{Th}}^{\mathrm{m}}$ (Kozai) $\mathcal{F}^\pm$ orientable,

$1 \neq$ e. value $\varphi_* : H^1(S) \circlearrowleft$

Then M^3_φ − (singular) has hyperbolic structures $\longrightarrow$ HP structures

= lifts of $\mathbb{H}^2$ reps $\rho^\pm \sim \tilde{\mathcal{F}}^\pm$

$$\begin{pmatrix} \multicolumn{3}{c}{SO(2,1)} & \begin{matrix} 0 \\ 0 \\ 0 \end{matrix} \\ \multicolumn{3}{c}{v^T} & 1 \end{pmatrix} \quad HP$$

$$\downarrow$$

$$\begin{pmatrix} \boxed{SO(1,1)} & \begin{matrix} 0 & 0 \\ 0 & 0 \end{matrix} \\ \begin{matrix} a & a \\ b & -b \end{matrix} & \begin{matrix} 1 & 0 \\ 0 & 1 \end{matrix} \end{pmatrix} \quad SOL$$

Idea: Smoothness of $SO(2,1)$ variety

at $\rho^\pm \Rightarrow \exists \rho_s : \longrightarrow SO(3,1)$

limit to $\tilde{\rho}_\pm \longrightarrow \mathrm{Isom}\, HP$

$$HP \xrightarrow[\text{transition}]{} Sol$$

Sol structure $\Rightarrow$ HP structure