Stochastic Computation

– A Brief Introduction –

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Computational Stochastics

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Objectives

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(A) algorithms for computational problems arising in stochastics.

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Example: Computation of probabilities, expectations, etc.

in complex stochastic models.

Applications in fluid dynamics, statistical physics, finance, etc.

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Question: When and how to use a random number generator for problems from analysis, optimization, stochastics, etc.

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Note: Computational tools vs. computational problems.

In the sequel, for illustration:

infinite-dimensional integration for stochastic differential equations.

Stochastic Differential Equations

Consider an autonomous, scalar SDE

$$dX_t = \mu(X_t)\,dt + \sigma(X_t)\,dW_t, \qquad t \in [0,T],$$
$$X_0 = x_0$$

with

- a scalar Brownian motion $W = (W_t)_{t\in[0,T]}$,
- drift and diffusion coefficients $\mu, \sigma : \mathbb{R} \to \mathbb{R}$,
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Assumption

$$\mu, \sigma \in \mathrm{Lip}_1(\mathbb{R}, \mathbb{R}).$$

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The solution $X = (X_t)_{t\in[0,T]}$ is a Markov process with continuous paths, and

$$\lim_{h\to 0} h^{-1} \cdot \mathrm{E}(X_{t+h} - X_t \,|\, X_t = x) = \mu(x),$$
$$\lim_{h\to 0} h^{-1/2} \cdot \big(\mathrm{E}((X_{t+h} - X_t)^2 \,|\, X_t = x)\big)^{1/2} = |\sigma|(x).$$

The Euler Scheme

The Euler scheme with time-discretization

$$t_k = t_k^K = k/K \cdot T$$

for $k = 0, \ldots, K$ is defined by

$$\begin{aligned}
\hat{X}_{t_0} &= x_0, \\
\hat{X}_{t_{k+1}} &= \hat{X}_{t_k} + \mu(\hat{X}_{t_k}) \cdot (t_{k+1} - t_k) \\
&\quad + \sigma(\hat{X}_{t_k}) \cdot (\underbrace{W_{t_{k+1}} - W_{t_k}}_{\text{iid, } N(0,T/K)}),
\end{aligned}$$

and piecewise linear interpolation.

This yields a stochastic process $\hat{X} = \hat{X}^K = (\hat{X}_t)_{t \in [0,T]}$.

The Computational Problem

Given $x_0 = 1$, μ, σ, and

$$f : C([0,T]) \to \mathbb{R},$$

compute

$$S(\mu, \sigma, f) = \mathrm{E}(f(X)) = \int_{C([0,T])} f \, dP_X .$$

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$\hookrightarrow$ Infinite-dimensional integration w.r.t. a measure that is given only implicitly.

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Assumption: $(\mu, \sigma, f) \in F$ for

$$F = \mathrm{Lip}_1(\mathbb{R}, \mathbb{R}) \times \mathrm{Lip}_1(\mathbb{R}, \mathbb{R}) \times \mathrm{Lip}_1(C([0,T]), \mathbb{R}).$$

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Monte Carlo Euler, with independent copies $\hat{X}_1^K, \ldots, \hat{X}_M^K$ of $\hat{X}^K$,

$$S^{K,M}(\mu, \sigma, f) = 1/M \cdot \sum_{m=1}^{M} f(\hat{X}_m^K)$$

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Questions:

- Quality of $S^{K,M}$?
- Are there better algorithms?

Resources

- real number machine; cost one per operation,
- ideal random number generator; cost one per call,
- oracles for $\mu(x)$ and $\sigma(x)$ for any $x \in \mathbb{R}$; cost one per call,
- oracle for $f(x)$ for any piecewise linear function $x \in C([0, T])$ with equidistant breakpoints; cost $=$ $\#$breakpoints $+ 2$.

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Maximal error and cost of $\mathcal{A}$

$$\mathrm{error}(\mathcal{A}) = \sup_{(\mu,\sigma,f)\in F} \left(\mathrm{E}(S(\mu,\sigma,f) - \mathcal{A}(\mu,\sigma,f))^2\right)^{1/2},$$

$$\mathrm{cost}(\mathcal{A}) = \sup_{(\mu,\sigma,f)\in F} \mathrm{E}(\mathrm{cost}(\mathcal{A}, (\mu,\sigma,f), \cdot)).$$

Results

Thm. With suitable parameters, Monte Carlo Euler $\mathcal{A}$ achieves

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See Heinrich (1998), Giles (2008), ...

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Thm. For the n-th minimal error

$$e(n) = \inf\{\mathrm{error}(\mathcal{A}) : \mathcal{A} \text{ randomized alg. with } \mathrm{cost}(\mathcal{A}) \leq n\}$$

we have

$$e(n) \succeq_{\log} n^{-1/2}.$$

See Creutzig, Dereich, Müller-Gronbach, R (2008).