Spatiotemporal trade-off for quasi-uniform sampling of evolving signals

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joint work with Akram Aldroubi and Ilya Krishtal

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The General Problem

Consider an operator $B : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ and sampling sets $\{\Lambda_k\}_{k=1}^{m}$, with $\Lambda_k \subset \mathbb{Z}$. Can we recover $x \in \ell^2(\mathbb{Z})$ from the samples $\left\{x|_{\Lambda_1}, (Bx)|_{\Lambda_2}, \ldots, (B^{m-1}x)|_{\Lambda_m}\right\}$?

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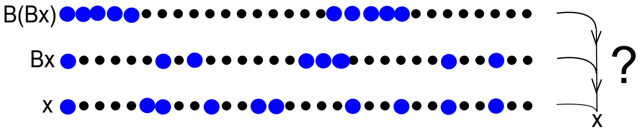

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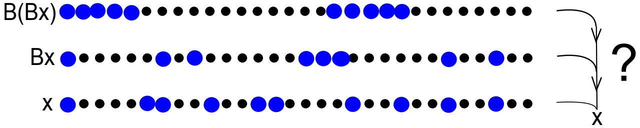
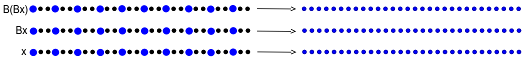


Inspired by the work of Lu, Vetterli, and their collaborators on spatio-temporal sampling of heat distributions.

Relation to existing fields

Sampling theory:

- When and how can we reconstruct a signal from samples of it?



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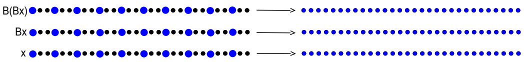

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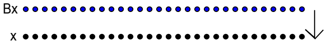
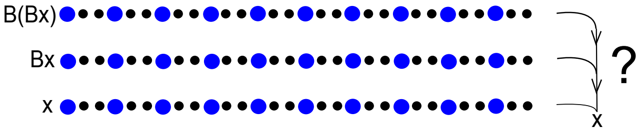
Inverse problems:

- How can we recover x from knowledge of Bx?



- We undo multiple iterations of B at once from partial knowledge of the signal at each iteration.
- We do not require the operator B to have a bounded inverse.

A "Simple" Dynamical Sampling Problem



Suppose B is a convolution operator, i.e, $Bx = a * x$ for some $a \in \ell^2(\mathbb{Z})$.

Consider only regular subsampling by a factor of m, i.e., $\Lambda_k = m\mathbb{Z}$ for all $k = 0, \ldots m-1$.

Regular Subsampling

Proposition

Suppose $Bx = a * x$ *for some* $a \in \ell^2(\mathbb{Z})$ *such that* $\hat{a} \in L^\infty(\mathbb{T})$. *Let* $S_m : \ell^2(\mathbb{Z}) \to \ell^2(\mathbb{Z})$ *denote the operator of subsampling by a factor of* m *so that* $(S_m z)(k) = z(mk)$ *and* $y_n = S_m((\underbrace{a * \ldots * a}_{n-1}) * x)$. *Define*

$$\mathcal{C}_m(\xi) = \begin{pmatrix} 1 & 1 & \ldots & 1 \\ \hat{a}(\frac{\xi}{m}) & \hat{a}(\frac{\xi+1}{m}) & \ldots & \hat{a}(\frac{\xi+m-1}{m}) \\ \vdots & \vdots & \vdots & \vdots \\ \hat{a}^{(m-1)}(\frac{\xi}{m}) & \hat{a}^{(m-1)}(\frac{\xi+1}{m}) & \ldots & \hat{a}^{(m-1)}(\frac{\xi+m-1}{m}) \end{pmatrix}, \quad (1)$$

$\xi \in \mathbb{T}$. *Then a vector* $x \in \ell^2(\mathbb{Z})$ *can be recovered in a stable way from the measurements* $y_n, n = 1, \ldots, m$, *i.e. the reconstruction operator is bounded, if and only if there exists* $\alpha > 0$ *such that the set* $\{\xi : |\det \mathcal{C}_m(\xi)| < \alpha\}$ *has zero measure.*

Additional Samples

When the previous proposition fails, we take additional samples by shifting and then subsampling by a large factor. Let T_c be the shift operator.

Theorem

Let $m \in \mathbb{Z}^+$ be fixed. Suppose that $\hat{a}$ is continuous and that $\mathcal{C}_m(\xi)$ is singular only when $\xi \in \{\xi_i\}_{i \in I}$. Suppose n is a positive integer such that $|\xi_i - \xi_j| \neq \frac{k}{n}$ for any $i, j \in I$ and $k \in \{1, \ldots, n-1\}$. Then the extra samples given by $\{(S_{mn}T_c)x\}_{c \in \{1, \ldots, m-1\}}$ provide enough additional information to stably recover any $x \in \ell^2(\mathbb{Z})$, i.e. the reconstruction operator is bounded.

- This means we need to choose n so that for any translation of the $\frac{1}{n}$-grid contains at most one singularity of $\mathcal{C}_m(\xi)$.
- If $|I| < \infty$, an n satisfying these conditions can always be found.

Example with a low pass filter

Theorem (Aldroubi, D, Krishtal)

Suppose $\hat{a}$ is real, symmetric, continuous, and strictly decreasing on $(0, \frac{1}{2})$. Then the matrix $\mathcal{C}_m(\xi)$ is singular only when $\xi = 0, \frac{1}{2}$. Then for any odd integer n, the extra samples given by $\{(S_{mn}T_c)x\}_{c \in \{1,\dots,\frac{m-1}{2}\}}$ provide enough additional information to stably recover any $x \in \ell^2(\mathbb{Z})$, i.e. the reconstruction operator is bounded.

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- By the previous theorem, we need to choose n so that $|0 - \frac{1}{2}| = \frac{1}{2}$ does not lie on the $\frac{1}{n}$-grid.

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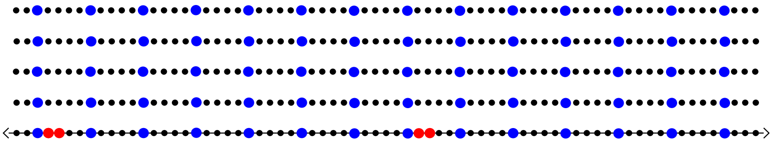
Example with $m = 5$, and $n = 7$.



Additional Time Samples

By taking additional samples at every time level, we are able to reduce the number of additional spatial samples required for stable reconstruction.

Theorem

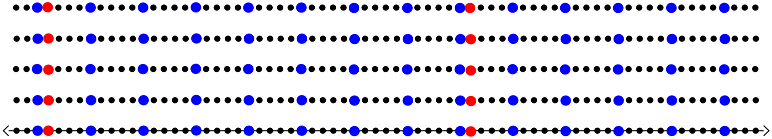
Let $p = \max_{\xi \in \mathbb{T}}\{\max$ *number of columns of* $\mathcal{C}_m(\xi)$ *that coincide*$\}$. *And suppose* n *satisfies the conditions of previous theorem. Let* $c_1, \ldots, c_p$ *be such that* $c_1 = 1 \mod m, c_2 = 2 \mod m, \ldots, c_p = p \mod m$. *Then the extra samples given by* $\{S_{mn}T_c(a^j * x)\}_{c \in \{c_1,\ldots,c_p\}, j=0,\ldots,m-1}$ *provide enough additional information to stably recover any* $x \in \ell^2(\mathbb{Z})$, *i.e. the reconstruction operator is bounded.*

Additional Time Samples with Low Pass Filter

Theorem

Let $m \in \mathbb{Z}^+$ be odd. Suppose $\hat{a}$ is real, symmetric, continuous, and decreasing on $(0, \frac{1}{2})$. Then for any odd n and any c relatively prime to m, the extra samples given by $\{(S_{mn}T_c)(a^j * x)\}_{j\in\{0,\dots,m-1\}}$ provide enough additional information to stably recover any $x \in \ell^2(\mathbb{Z})$, i.e. the reconstruction operator is bounded.

Example with $m = 5$, $n = 7$, $c = 1$.



Bounds on the Norm of the Reconstruction Operator

Theorem (Aldroubi, D, Krishtal)

Let $\mathbf{A}$ *be the dynamical sampling operator with additional sampling given by* $\{S_{mn}(T_c x)\}_{c=1}^{m-1}$,
where $\hat{a}$ *is real, symmetric, continuous, and decreasing on* $(0, \frac{1}{2})$,
and the derivative of $\hat{a}$ *is continuous and nonzero on* $(0, \frac{1}{2})$.
Then the reconstruction operator, $\mathbf{A}^\dagger$, *is bounded above by*

$$\|\mathbf{A}^\dagger\| \leq m\beta_1(1 + m\sqrt{n-1}) \tag{2}$$

where $\beta_1 = \max\{n, \operatorname*{ess\,sup}_{\xi \in J} \|\mathcal{C}_m^{-1}(\xi)\|\} < \infty$, *and*
$J = [\frac{1}{4n}, \frac{1}{2} - \frac{1}{4n}] \cup [\frac{1}{2} + \frac{1}{4n}, 1 - \frac{1}{4n}]$,

Conclusion

Summary

- We provide theoritical results for finding stable spatio-temporal sampling sets for recovery of signals in evolutionary systems goverened by a convolution operator.
- We show that by exploiting the structure of the system, time samples can be traded for spatial samples. Thus, reducing the number of spatial samples needed to recover the signal.

Future Work

- How can we use regularization to improve the bound of the reconstruction operator?
- What about other sampling schemes and other types of operators?