Relationship of discrepancy theory with harmonic analysis, approximation theory, and probability

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Uniform Distribution Theory and Applications

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The small ball inequality

Small Ball Conjecture

For dimensions $d \geq 2$, we have

$$n^{\frac{d-2}{2}} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

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- Known: $\frac{d-1}{2} + \eta(d)$ for $d \geq 3$ (DB, Lacey, Vagharshakyan, 2008)

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$$\Big\| \sum_{|R|=2^{-n}} \varepsilon_R h_R \Big\|_2 \approx \left(\#\mathbb{H}_n^d \cdot 2^n \cdot 1 \cdot 2^{-n} \right)^{\frac{1}{2}} \approx n^{\frac{d-1}{2}}$$

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Sidon's theorem

If a bounded 2π-periodic function f has lacunary Fourier series $\sum_{k=1}^{\infty} a_k e^{in_k x}$, $n_{k+1}/n_k > \lambda > 1$, then

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- Riesz product: $P_K(x) = \prod_{k=1}^{K} (1 + \varepsilon_k \cos n_k x)$

Small Ball Problem for the Brownian Sheet

Let $B : [0,1]^d \longrightarrow \mathbb{R}$ be the Brownian Sheet, i.e. a centered Gaussian process with covariance $\mathbb{E}B(s)B(t) = \prod_{k=1}^{d} \min\{s_k, t_k\}$.

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- Lower bound is known in $d = 2$ (Talagrand)

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$$\mathbb{P}(\|B\|_\infty < \epsilon) \leq \mathbb{P}\left(\Big\| \sum_{|R|=2^{-n}} g_R \eta_R \Big\|_\infty < \epsilon \right)$$
$$\leq \mathbb{P}\left(2^{-3n/2} n^{-\frac{1}{2}(d-2)} \sum_{|R|=2^{-n}} |g_R| < \epsilon \right)$$

Metric entropy of mixed smoothness classes

- Let $T : L^p([0,1]^d) \to C([0,1]^d)$ be the integration operator: $(Tf)(x) = \int_0^{x_1} ... \int_0^{x_d} f(y)dy$.

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$-\log \mathbb{P}(\|B\|_{C[0,1]^d} < \epsilon) \approx \epsilon^{-2} \left(\log \frac{1}{\epsilon}\right)^\beta \quad iff$

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Conjecture

For $d \geq 2$, one has the estimate $\log N_2(\epsilon) \gtrsim \frac{1}{\epsilon} \left(\log \frac{1}{\epsilon}\right)^{d-1/2}$

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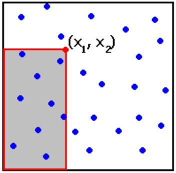
- One can choose *many* σ's for which this sum is *large* (Varshamov-Gilbert bound)

Discrepancy function

Consider a set $\mathcal{P}_N \subset [0,1]^d$ consisting of N points:



Define the discrepancy function of the set $\mathcal{P}_N$ as

$$D_N(x) = \sharp\{\mathcal{P}_N \cap [0,x)\} - N x_1 x_2 \dots x_d$$

Theorem (Roth, 1954 ($p = 2$); Schmidt, 1977 ($1 < p < 2$))

The following estimate holds for all $\mathcal{P}_N \subset [0,1]^d$ with $\#\mathcal{P}_N = N$:

$$\|D_N\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

L^p estimates, $1 < p < \infty$

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Theorem (Davenport, 1956 ($d = 2$, $p = 2$); Roth, 1979 ($d \geq 3$, $p = 2$); Chen, 1983 ($p > 2$); Chen, Skriganov, 2000's)

There exist sets $\mathcal{P}_N \subset [0,1]^d$ *with*

$$\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$$

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- Main idea:

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- For a proper choice of signs ε_R

$$\langle D_N, f_{\vec{r}}\rangle \gtrsim 1$$

Roth's method

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- $\langle D_N, F \rangle \gtrsim \# \mathbb{H}_n^d \approx n^{d-1} \approx (\log N)^{d-1}.$
- Thus
$$\|D_N\|_2 \geq \frac{\langle D_N, F \rangle}{\|F\|_2} \gtrsim (\log N)^{\frac{1}{2}}$$

Littlewood-Paley inequalities

- Let $f = \sum_{I \in \mathcal{D}} a_I h_I$

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- Sharp constants (Wang, '91): $B_p \approx \sqrt{p}$

Product Littlewood-Paley inequalities

- Let $f = \sum_{R\in\mathcal{D}^d} a_R h_R$
- The *dyadic* product Littlewood-Paley square function

$$S_d f = \Big(\sum_{I\in\mathcal{D}} |a_R|^2 \mathbf{1}_R \Big)^{1/2}$$

Theorem

For all $1 < p < \infty$, there exist A_p, $B_p > 0$ such that

$$A_p^d \|S_d f\|_p \le \|f\|_p \le B_p^d \|S_d f\|_p$$

- R. Fefferman, J. Pipher

L^∞ estimates

Conjecture

$$\|D_N\|_\infty \gg (\log N)^{\frac{d-1}{2}}$$

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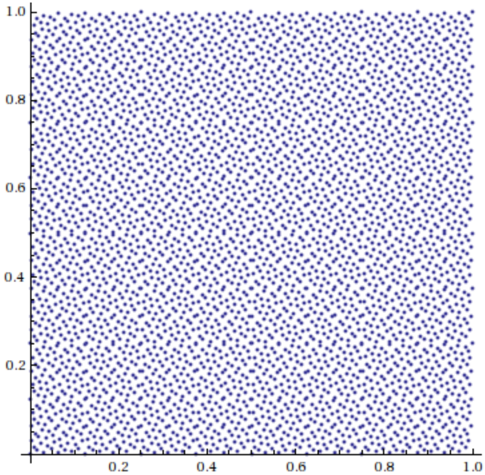
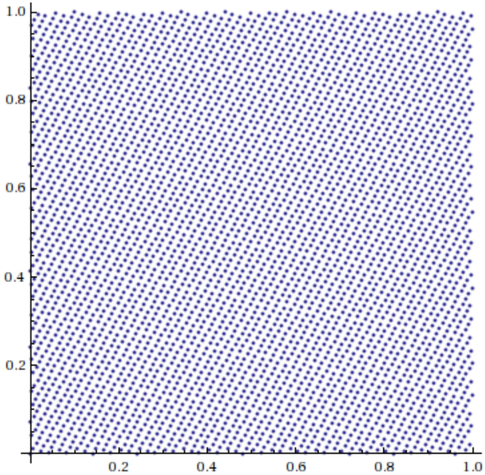
Low discrepancy sets



The van der Corput set with $N = 2^n$ points (here $n = 12$)

$$\big(0.x_1x_2...x_n,\ 0.x_nx_{n-1}...x_2x_1\big), \quad x_k = 0 \text{ or } 1.$$

Discrepancy $\approx \log N$



The irrational ($\alpha = \sqrt{2}$) lattice with $N = 2^{12}$ points

$$\left(n/N, \{n\alpha\}\right), \quad n = 0, 1, ..., N-1.$$

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$d \geq 3$, Halton, Hammersley (1960):

There exist $\mathcal{P}_N \subset [0,1]^d$ with $\|D_N\|_\infty \lesssim (\log N)^{d-1}$

Conjectures and results

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$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

Theorem (Roth, 1954)

In all dimensions $d \geq 2$ the following estimate holds for all N-point distributions $\mathcal{P}_N \subset [0,1]^d$:

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}} .$$

Conjectures and results

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$\|D_N\|_\infty \gtrsim (\log N)^{d-1}$

Conjecture 2

$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

Theorem (Beck, 1989)

In dimension $d = 3$ the following estimate holds for all N-point distributions $\mathcal{P}_N \subset [0,1]^3$:

$$\|D_N\|_\infty \gtrsim (\log N) \cdot (\log \log N)^{\frac{1}{8} - \varepsilon} .$$

Conjectures and results

Conjecture 1

$\|D_N\|_\infty \gtrsim (\log N)^{d-1}$

Conjecture 2

$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$

Theorem (DB, Lacey ($d = 3$); DB, Lacey, Vagharshakyan ($d > 3$), 2008)

For $d \geq 3$ there exists $\eta > 0$ such that the following estimate holds for all N-point distributions $\mathcal{P}_N \subset [0,1]^d$:

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}.$$

Near L^∞: BMO and $\exp(L^\alpha)$ estimates

Theorem (DB, Lacey, Parissis, Vagharshakyan ($d = 2$), 2009; DB, Markhasin, ($d \geq 3$) 2014)

- *For any N-point set $\mathcal{P}_N \subset [0,1]^d$ we have*

$$\|D_N\|_{\mathrm{BMO}} \gtrsim (\log N)^{\frac{d-1}{2}}$$

- *There exist sets that satisfy*

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Theorem (DB, Lacey, Parissis, Vagharshakyan, 2009)

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$$\|D_N\|_{\exp(L^\alpha)} \gtrsim (\log N)^{1-1/\alpha}, \qquad 2 \leq \alpha < \infty .$$

- *"Digit shifts" of the van der Corput set satisfies*

$$\|D_N\|_{\exp(L^\alpha)} \lesssim (\log N)^{1-1/\alpha}, \qquad 2 \leq \alpha < \infty .$$

Exponential estimates in higher dimensions

Theorem (Skriganov, 2014; DB, Markhasin, 2014)

There exist sets $\mathcal{P}_N \subset [0,1]^d$ ("random digit shifts of digital nets" or "higher order nets") for which

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- This would imply that

$$\mu\{x : D_N(x) \geq (\log N)^{d/2}\} \lesssim N^{-c}.$$

Other endpoint: L^1

Theorem (Halász, 1981)

In dimension $d = 2$ for any collection of N points $\mathcal{P}_N \subset [0,1]^2$

$$\|D_N\|_1 \gtrsim \sqrt{\log N}.$$

- $C_1 \geq 0.00854...$ (Vagharshakyan, 2013)
- This continues to hold for $d \geq 3$: $\|D_N\|_1 \gtrsim \sqrt{\log N}$
- ... nothing better is known in higher dimensions!
- Conjecture:

$$\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$$

Discrepancy function $D_N(x) = \#\{\mathcal{P}_N \cap [0,x)\} - Nx_1x_2$	**Lacunary Fourier series** $f(x) \sim \sum_{k=1}^{\infty} c_k \sin n_k x$, $\frac{n_{k+1}}{n_k} > \lambda > 1$
$\|D_N\|_2 \gtrsim \sqrt{\log N}$ (Roth, '54)	$\|f\|_2 \equiv \sqrt{\sum \|c_k\|^2}$
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81) Riesz product: $\prod (1 + cf_k)$	$\|f\|_\infty \gtrsim \sum \|c_k\|$ (Sidon, '27) Riesz product: $\prod (1 + \cos(n_k x + \phi_k))$
$\|D_N\|_1 \gtrsim \sqrt{\log N}$ (Halász, '81) Riesz product: $\prod \left(1 + i \cdot \frac{c}{\sqrt{\log N}} f_k\right)$	$\|f\|_1 \gtrsim \|f\|_2$ (Sidon, '30) Riesz product: $\prod \left(1 + i \cdot \frac{\|c_k\|}{\|f\|_2} \cos(n_k x + \theta_k)\right)$

Table: Discrepancy function and lacunary Fourier series

Estimates "near" L^1

Theorem (Lacey, 2010)

$$\|D_N\|_{L(\log L)^{\frac{d-2}{2}}} \gtrsim (\log N)^{\frac{d-1}{2}}.$$

- $L(\log L)^{\frac{d-1}{2}}$ is "easy"

Theorem (Lacey, 2010)

For $0 < p \leq 1$ we have the estimate in the (dyadic) d-parameter Hardy space

$$\|D_N\|_{H^p} \gtrsim (\log N)^{\frac{d-1}{2}}.$$

Other endpoint: L^1

- Conjecture: $\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$
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Theorem (Amirkhanyan, DB, Lacey, 2013: L^1 "dichotomy")

- *If* $\mathcal{P}_N \subset [0,1]^d$ *satisfies* $\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$ *then*
$$\|D_N\|_1 \gtrsim (\log N)^{\frac{d-1}{2}}$$
- *Every* $\mathcal{P}_N \subset [0,1]^d$ *satisfies either*
$$\|D_N\|_1 \geq (\log N)^{(d-1)/2-\epsilon} \quad \text{or} \quad \|D_N\|_2 \geq \exp(c(\log N)^{\epsilon}) \, .$$
- *For* $d \geq 3$*, if* $\|D_N\|_1 \lesssim \sqrt{\log N}$*, then* $\|D_N\|_2 \gtrsim N^C$.

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Theorem (Amirkhanyan, DB, Lacey, 2013: L^1 "dichotomy")

- *If $\mathcal{P}_N \subset [0,1]^d$ satisfies $\|D_N\|_p \lesssim (\log N)^{\frac{d-1}{2}}$ then*
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- *For $d \geq 3$, if $\|D_N\|_1 \lesssim \sqrt{\log N}$, then $\|D_N\|_2 \gtrsim N^C$.*

Theorem (Skriganov 2014)

Let $0 < p \leq 1$. For any set $\mathcal{P}_N$ there exists a digit shift T with
$$\|D_N(\mathcal{P}_N \oplus T)\|_p \gtrsim (\log N)^{\frac{d-1}{2}}$$

Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of α_R

$$n^{\frac{1}{2}(d-2)} \Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim 2^{-n} \sum_{|R|=2^{-n}} |\alpha_R|$$

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Conjecture 2

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

The small ball conjecture and discrepancy

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Conjecture 2

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

- In both conjectures one gains a square root over the L^2 estimate.

The small ball conjecture and discrepancy

Signed Small Ball Conjecture

For dimensions $d \geq 2$, we have for all choices of $\alpha_R = \pm 1$

$$\Big\| \sum_{|R|=2^{-n}} \alpha_R h_R \Big\|_\infty \gtrsim n^{\frac{d}{2}}$$

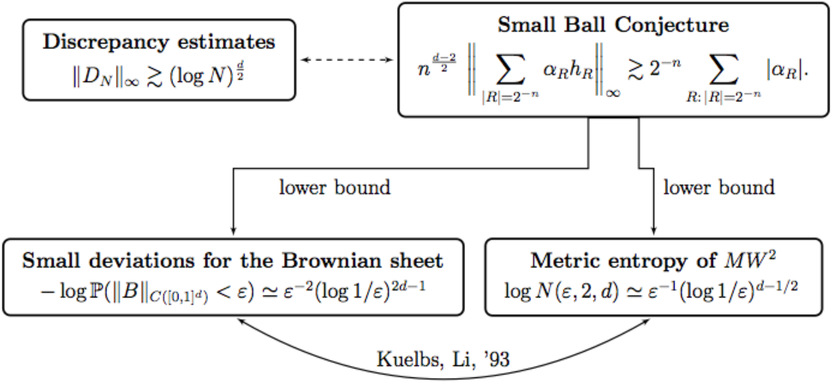
Conjecture 2

$$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d}{2}}$$

- In both conjectures one gains a square root over the L^2 estimate.

Discrepancy estimates	**Small Ball inequality (signed)**
Dimension $d = 2$	
$\|D_N\|_\infty \gtrsim \log N$ (Schmidt, '72; Halász, '81)	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n$ (Talagrand, '94; Temlyakov, '95)
Higher dimensions, L^2 bounds	
$\|D_N\|_2 \gtrsim (\log N)^{(d-1)/2}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_2 \gtrsim n^{(d-1)/2}$
Higher dimensions, conjecture	
$\|D_N\|_\infty \gtrsim (\log N)^{d/2}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{d/2}$
Higher dimensions, known results	
$\|D_N\|_\infty \gtrsim (\log N)^{\frac{d-1}{2}+\eta}$	$\Big\| \sum_{\|R\|=2^{-n}} \varepsilon_R h_R \Big\|_\infty \gtrsim n^{\frac{d-1}{2}+\eta}$

Connections between problems

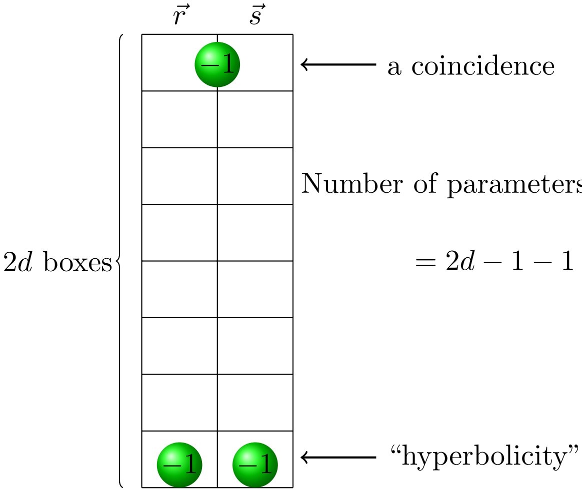


"Beck Gain" lemma: preservation of orthogonality

Lemma

Beck Gain: We have the estimate

$$\Big\| \sum_{\substack{\vec{r} \neq \vec{s} \in \mathbb{H}_n^d \\ r_1 = s_1}} f_{\vec{r}} \cdot f_{\vec{s}} \Big\|_p \lesssim p^{(2d-1)/2} n^{(2d-3)/2}$$



Number of parameters

$$= 2d - 1 - 1 - 1$$

Lower and upper bounds in dimension $d = 2$

	LOWER BOUND	UPPER BOUND
	Axis-parallel rectangles	
L^∞	$\log N$	$\log N$
L^2	$\log^{\frac{1}{2}} N$	$\log^{\frac{1}{2}} N$
	Rotated rectangles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Circles	
	$N^{1/4}$	$N^{1/4}\sqrt{\log N}$
	Convex Sets	
	$N^{1/3}$	$N^{1/3}\log^4 N$

Higher dimensions: $d \geq 3$

LOWER BOUND		UPPER BOUND
Axis-parallel boxes		
L^∞	$(\log N)^{\frac{d-1}{2}+\eta}$	$(\log N)^{d-1}$
L^2	$(\log N)^{\frac{d-1}{2}}$	$(\log N)^{\frac{d-1}{2}}$
Rotated boxes		
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
Balls		
	$N^{\frac{1}{2}-\frac{1}{2d}}$	$N^{\frac{1}{2}-\frac{1}{2d}}\sqrt{\log N}$
Convex Sets		
	$N^{1-\frac{2}{d+1}}$	$N^{1-\frac{2}{d+1}}\log^c N$