Two added structures in sparse recovery: nonnegativity and disjointedness

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Semester Program on "High-Dimensional Approximation"
ICERM
7 October 2014

Part I:
Nonnegative Sparse Recovery

(joint work with D. Koslicki)

Motivation from Metagenomics

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- Codes available at
  `sourceforge.net/projects/quikr/`
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Morale: ℓ_1-minimization not suited for nonnegative sparse recovery.

Nonnegative Least Squares

- To solve the feasibility problem, one may consider

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- Connection with OMP explains suitability for sparse recovery.

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- For frequency matrices, as $\lambda \to \infty$, the minimizer $\mathbf{x}_\lambda$ tends to the minimizer of $\|\mathbf{z}\|_1$ subject to $\mathbf{A}\mathbf{z} = \mathbf{y}$ and $\mathbf{z} \geq \mathbf{0}$.

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- For Gaussian matrices (RNSP and QP hold), the solutions $\mathbf{x}_\lambda$ of (REG) with $\mathbf{y} = \mathbf{A}\mathbf{x} + \mathbf{e}$ obey, for all $\mathbf{x} \in \mathbb{R}^N$ and $\mathbf{e} \in \mathbb{R}^m$,

$$\|\mathbf{x} - \mathbf{x}_\lambda\|_1 \leq C\, \sigma_s(\mathbf{x})_1 + D\sqrt{s}\, \|\mathbf{e}\|_2 + \frac{E\, s}{\lambda^2} \|\mathbf{x}\|_1.$$

Part II:
Disjointed Sparse Recovery

(joint work with M. Minner and T. Needham)

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  - there are $\geq d$ zero entries between two nonzero entries.

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$$m_{\rm spa\&dis} \asymp s \ln\left(e\frac{N-d(s-1)}{s}\right).$$
- There is no benefit in knowing the simultaneity of sparsity and disjointedness over knowing only one of the structures, since
$$m_{\rm spa\&dis} \asymp \min\left\{m_{\rm spa}, m_{\rm dis}\right\}.$$

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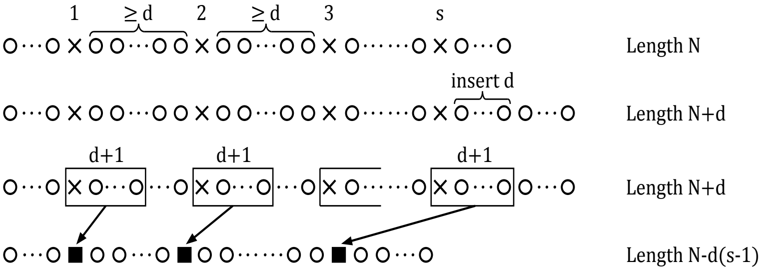

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- The adaptation of iterative hard thresholding is

$$\mathbf{x}^{n+1} = \mathbf{P}_{s,d}(\mathbf{x}^n + \mathbf{A}^*(\mathbf{y} - \mathbf{A}\mathbf{x}^n)),$$

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- For any s-sparse d-disjointed $\mathbf{x} \in \mathbb{R}^N$ and any $\mathbf{e} \in \mathbb{R}^m$,

$$\|\mathbf{x} - \lim_{n\to\infty} \mathbf{x}^n\|_2 \leq D\|\mathbf{e}\|_2$$

  as soon as the RI-like property

$$(1-\delta)\|\mathbf{z}+\mathbf{z}'+\mathbf{z}''\|_2^2 \leq \|\mathbf{A}(\mathbf{z}+\mathbf{z}'+\mathbf{z}'')\|_2^2 \leq (1+\delta)\|\mathbf{z}+\mathbf{z}'+\mathbf{z}''\|_2^2$$

  holds with $\delta < 1/2$ for all s-sparse d-disjointed $\mathbf{z}, \mathbf{z}', \mathbf{z}'' \in \mathbb{R}^N$.

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- Similar results obtained earlier for the adaptation of CoSaMP.

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- A dynamic program can be solved in $\mathcal{O}(N^2)$ operations.
- Determine $F(N,s)$, where

$$F(n,r) := \min\left\{\sum_{j=1}^{n} |x_j - z_j|^2 : \mathbf{z} \in \mathbb{C}^n \text{ } r\text{-sparse } d\text{-disjointed}\right\}$$

$$= \min\left\{\begin{array}{c} F(n-1,r) + |x_n|^p, \\ F(n-d-1,r-1) + \sum\limits_{j=n-d}^{n-1} |x_j|^p. \end{array}\right.$$

Computing the Projection $\mathbf{P}_{s,d}$, ctd.

Dynamic program for $\mathbf{x} = (1, 0, 1, 2^{1/4}, 1, 0, 2^{-1/2})$, $s = 3$, $d = 1$.

$\mathbf{x}$
1
0
1
1.1892
1
0
0.7071

$F(n,r)$	$r=0$	$r=1$	$r=2$	$r=3$
$n=1$	*1*	*0*	*0*	*0*
$n=2$	*1*	*0*	*0*	*0*
$n=3$	*2*	1	0	0
$n=4$	*3.4142*	2	1	1
$n=5$	*4.4142*	3	2	1.4142
$n=6$	*4.4142*	3	2	1.4142
$n=7$	*4.9142*	3.5	2.5	1.9142

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- Noninflating measurements relative to our model:

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- Δ reconstruction map providing the robust estimate

$$\|\mathbf{x} - \Delta(\mathbf{Ax} + \mathbf{e})\|_2 \leq D\|\mathbf{e}\|_2,$$

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- Then

$$m \ge C\, s \ln\left(e\frac{N - d(s-1)}{s}\right).$$

Key Combinatorial Lemma

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There exist

$$n \geq \left(\frac{N - d(s-1)}{c_1 s} \right)^{c_2 s}$$

d-disjointed subsets $S_1, \ldots, S_n$ of $[\![1 : N]\!]$ such that

$$\mathrm{card}(S_i) = s \quad \text{for all } i, \qquad \mathrm{card}(S_i \cap S_j) < \frac{s}{2} \quad \text{for all } i \neq j.$$

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This extends a crucial result known for $d = 0$ (sparse vectors), but the counting argument must be somewhat refined.