Constructive tractability of the Helmholtz problem

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Helmholtz problem

For

$$f \in F_d, q \in Q_d,$$

find an approximation of the solution $u = u_{f,q}$ to

$$L_q u := -\Delta u + qu = f \qquad \text{in } I^d := (0,1)^d,$$

with either Dirichlet

$$u = g \qquad \text{on } \partial I^d$$

or Neumann

$$\partial_\nu u = g \qquad \text{on } \partial I^d$$

boundary conditions.
Actually consider variational form of this problem ($H^1(I^d)$-error).

Here, d can be huge!!!

Our previous work

- "Tractability of quasilinear problems I: general results", *J. Approx. Theory*, 2007.
- "Tractability of quasilinear problems II: second-order elliptic problems", *Math. Comp.*, 2007.
- "Tractability of multivariate approximation over a weighted unanchored Sobolev space", *Constr. Approx.*, 2009.
- "Tractability of the Helmholtz equation with non-homogeneous Neumann boundary conditions: relation to L_2-approximation", *J. Comp.*, 2009 (only W).
- "Tight tractability results for a model second-order Neumann problem", *J. FoCM.*, 2014.

What we really want

- General q.
- General Dirichlet or Neumann boundary conditions.
- Wide range of weighted spaces.
- Necessary and sufficient weight conditions for various flavors of tractability.
- Explicit optimal tractability algorithms.

What you're getting today

- General q, but sometimes specializing to $q \equiv 1$.
- Homogeneous Neumann boundary conditions.
- Variety of F_d and Q_d.
- Tractability results, but sometimes optimal tractability algorithms.

Problem definition

- Let

$$F_d = \text{unit ball of } H_{d,\gamma} = H(K_{d,\gamma}),$$
$$Q_d = \{\, q \in F_d : q(\cdot) \geq q_0 \,\}$$

where $q_0 \in (0, 1)$.

- Let

$$B_d(v, w; q) = \int_{I^d} [\nabla v \cdot \nabla w + qvw] \qquad \forall\, v, w \in H^1(I^d), q \in Q_d.$$

- Seek $u = S_d(f, q) \in H^1(I^d)$:

$$B_d(u, w; q) = \langle f, w \rangle_{L_2(I^d)} \qquad \forall\, w \in H^1(I^d),$$

for $(f, q) \in F_d \times Q_d$, i.e., the variational solution of

$$-\Delta u + qu = f \qquad \text{in } I^d,$$
$$\partial_\nu u = 0 \qquad \text{on } \partial I^d.$$

Quasilinearity

- S_d is *quasilinear*:
  - Linear in first argument.
  - Lipschitz in both arguments.
- See WW, 2007a.

“Usual IBC stuff”

- Continuous linear information.
- Error of algorithm A_n using at most n info evals:

$$e(A_n, S_d) = \sup_{[f,q]\in F_d\times Q_d} \|S_d(f,q) - A_n(f,q)\|_{H^1(I^d)}$$

- nth minimal error:

$$e(n, S_d) = \inf_{A_n} e(A_n, S_d)$$

- Info complexity:

$$n(\varepsilon, S_d) = \inf\{\, n \in \mathbb{N}_0 : e(n, S_d) \le \mathrm{CRI}_d\, \varepsilon \,\}$$

$$\mathrm{ABS} : \mathrm{CRI}_d \equiv 1 \qquad \mathrm{NOR} : \mathrm{CRI}_d = e(0, S_d)$$

Reduction to approximation problem

- Easy lower bound: $S_d \succcurlyeq \mathrm{APP}_{H_{d,\gamma} \to [H^1(I^d)]^*}$
- Known upper bound:
  - $S_d(\cdot, q)$ is $[H^1(I^d)]^*$-Lipschitz
  - $S_d(f, \cdot)$ is $L_2(I^d)$-Lipschitz
  - $S_d \preccurlyeq \mathrm{APP}_{H_{d,\gamma} \to L_2(I^d)}$
- New upper bound: S_d is $[H^1(I^d)]^*$-Lipschitz ... vile constants.
- Interpolatory algorithm $A_n(f, q) = S_d(\tilde{f}, \tilde{q})$
  - yields sharp info complexity bounds
  - no joy implementation-wise when $q \not\equiv$ const
- How to find easily-implementable "good" algorithms?
  We are currently investigating Galerkin algorithms, but we are not yet done ...

Tractability results for arbitrary $q \in Q_d$ [WW 2007]

- $H_{d,\boldsymbol{\gamma}} = H(K_{d,\boldsymbol{\gamma}})$, where

$$K_{d,\boldsymbol{\gamma}}(\mathbf{x},\mathbf{y}) = \sum_{\substack{\mathfrak{u}\subseteq\{1,2,\dots,d\}\\ |\mathfrak{u}|\le\omega}} \gamma_{d,\mathfrak{u}} \prod_{j\in\mathfrak{u}} K(x_j,y_j),$$

$$\text{with } \int_{[0,1]^2} K(x,y)\,dx\,dy \in (0,\infty)$$

- Finite-order weights

$$\gamma_{d,\mathfrak{u}} = 0 \qquad \text{for all } |\mathfrak{u}| > \omega$$

Tractability results for arbitrary $q \in Q_d$ [WW 2007]

- ABS + FOW $\Longrightarrow$ PT:

$$n(\varepsilon, S_d) \le C\varepsilon^{-2} d^{2\omega}$$

- ABS + FOW + $\sup_d \sum_{|\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} < \infty \implies$ SPT:

$$n(\varepsilon, S_d) \le C\varepsilon^{-2}$$

- NOR + FOW $\Longrightarrow$ PT:

$$n(\varepsilon, S_d) \le C\varepsilon^{-2} d^{\omega}$$

- NOR + FOW + $\sup_d \sum_{|\mathfrak{u}| \le \omega} \gamma_{d,\mathfrak{u}} < \infty \implies$ SPT:

$$n(\varepsilon, S_d) \le C\varepsilon^{-2}$$

- Also have results for Λ^{std}, as well as for Dirichlet problems.
- Based on Wasilkowski+W, 2004.

Tractability results for $q \equiv 1$ [WW 2014]

- Now S_d is linear!
- Here, F_d is unit ball of $H_{d,\boldsymbol{\gamma}}$.
  - Choose general weights (not necessarily FOW)

$$\boldsymbol{\gamma} = \{\, \gamma_{d,\mathfrak{u}} \geq 0 \,:\, \mathfrak{u} \subseteq [d] := \{1, 2, \ldots, d\},\ d \in \mathbb{N} \,\}$$

    with $\gamma_{d,\emptyset} = 1$.
  - Then

$$H_{d,\boldsymbol{\gamma}} = \{\, w \in [H^1(I)]^{\otimes d} :\ \partial_{\mathfrak{u}} w \equiv 0 \text{ whenever } \gamma_{d,\mathfrak{u}} = 0 \,\},$$

    with $\partial_{\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \partial_j$, where

$$\langle v, w \rangle_{H_{d,\boldsymbol{\gamma}}} = \sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{-1} \langle \partial_{\mathfrak{u}} v, \partial_{\mathfrak{u}} w \rangle_{L_2(I^d)} \qquad \forall\, v, w \in H_{d,\boldsymbol{\gamma}}.$$

- ABS=NOR, since $e(0, S_d) = 1$.

Tractability results for $q \equiv 1$ [WW 2014]

Everything depends on eigenpairs $(\beta_{\mathbf{k},\boldsymbol{\gamma}}, e_{\mathbf{k}})$ of $S_d^* S_d$ for $\mathbf{k} \in \mathbb{N}^d$:

- Eigenvalues:

$$\beta_{\mathbf{k},\boldsymbol{\gamma}} = \frac{1}{1 + \pi^2 \sum_{j=1}^{d} (k_j - 1)^2} \cdot \frac{1}{1 + \sum_{\emptyset \neq \mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} [\pi^2 (k_j - 1)^2]}$$

- Eigenvectors:

$$e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^{d} \cos[\pi(k_j - 1) x_i]$$

- Optimal algorithm:
  Let $n = |M(\varepsilon, d, \boldsymbol{\gamma})|$, where
  $M(\varepsilon, d, \boldsymbol{\gamma}) = \{\, \mathbf{k} \in \mathbb{N}^d : \beta_{\mathbf{k},\boldsymbol{\gamma}} > \varepsilon^2 \,\}$. Then

$$A_n(f, 1) = \sum_{\mathbf{k} \in M(\varepsilon, d, \boldsymbol{\gamma})} \frac{\langle f, e_{\mathbf{k}} \rangle_{H_{d,\gamma}}}{\|e_{\mathbf{k}}\|_{H_{d,\gamma}}^2} S_d e_{\mathbf{k}}$$

Tractability results for $q \equiv 1$ [WW 2014]

- Relation to L_2-approximation: Let $\mathrm{APP}_{d,\boldsymbol{\gamma}} = \mathrm{APP}_{H_{d,\gamma} \to L_2(I^d)}$ and

$$M_d = \max_{j=1,2,\ldots,d} \gamma_{d,\{j\}} < \infty \qquad \text{and} \qquad c_d = \min\{1, M_d^{-1}\}.$$

Then

$$n(\sqrt{\varepsilon} c_d^{-1/4}, \mathrm{APP}_{d,\boldsymbol{\gamma}}) \le n(\varepsilon, S_d) \le n(\varepsilon, \mathrm{APP}_{d,\boldsymbol{\gamma}}).$$

- $\mathrm{APP}_{d,\boldsymbol{\gamma}}$ was studied in WW, 2009.

Tractability for $q \equiv 1$ + product weights [WW 2014]

- Let $1 \geq \gamma_1 \geq \gamma_2 \geq \cdots > 0$. Then $\gamma_{d,\mathfrak{u}} = \prod_{j \in \mathfrak{u}} \gamma_j$.
- Quasi-polynomial tractability

$$n(\varepsilon, S_d) \leq C \exp\big(t(1 + \ln \varepsilon^{-1})(1 + \ln d)\big),$$

holds for all product weights, with

$$t = \frac{2}{\ln(1 + \pi^2)} \doteq 0.838233$$

- Polynomial tractability holds iff $\exists\, \tau > \frac{1}{2}$:

$$B_\tau = \frac{\zeta(2\tau)}{\pi^{2\tau}} \limsup_{d \to \infty} \frac{1}{\ln d} \sum_{j=1}^{d} \gamma_j^\tau < \infty,$$

in which case

$$n(\varepsilon, S_d) \leq C_\tau d^{B_\tau} \varepsilon^{-2\tau}.$$

Tractability for $q \equiv 1$ + product weights [WW 2014]

- Strong polynomial tractability iff $\exists\, \tau > \frac{1}{2}$:

$$A_\tau := \sup_{d \in \mathbb{N}} \sum_{j=1}^{d} \gamma_{d,j}^{\tau} < \infty,$$

When this holds, let

$$\tau^* = \inf\{\, \tau > \tfrac{1}{2} : A_\tau < \infty \,\}.$$

Then for all $\tau > \tau^*$, we have

$$n(\varepsilon, S_d) \leq \varepsilon^{-2\tau} \exp\left(\zeta(2\tau)\, \pi^{-2\tau}\, A_\tau\right)$$

Future work

- Study general q.
- We believe that results for $q \in Q_d$ will be analogous to those for $q \equiv 1$, but work is still in progress.

A new approximation problem

- For $s \in \mathbb{N}_0$, let
$$\|v\|^2_{H^s(H_{d,\gamma})} = \sum_{\mathfrak{u}\subseteq[d]} \gamma^{-1}_{d,\mathfrak{u}} \|\partial_{\mathfrak{u}} v\|^2_{H^s(I^d)}.$$
- $\mathrm{APP}_{r,s}$: Approximate $H^s(H_{d,\boldsymbol{\gamma}})$-functions in the $H^r(I^d)$-norm, where $r, s \in \mathbb{N}_0$ and $r \le s$.
- Then Galerkin error is bounded by $e(\mathrm{APP}_{1,2})$, modulo a constant, with still-unknown dependence on d.

Spectral results for $\mathrm{APP}_{r,s}$

- Let $W_{r,s} = \mathrm{APP}^*_{r,s}\,\mathrm{APP}_{r,s}$. Then $e(n, \mathrm{APP}_{r,s}) = \sqrt{\beta_{n+1,r,s}}$, where $\beta_{1,r,s} \geq \beta_{2,r,s} \geq \cdots > 0$ are eigenvalues of $W_{r,s}$.
- For $\mathbf{k} \in \mathbb{N}^d$, let $e_{\mathbf{k}}(\mathbf{x}) = \prod_{j=1}^d \cos[\pi(k_j - 1)x_j]$. Then

$$W_{r,s} e_{\mathbf{k}} = \beta_{r,s,\boldsymbol{\gamma},\mathbf{k}} e_{\mathbf{k}} \qquad \forall\, \mathbf{k} \in \mathbb{N}^d,$$

where

$$\beta_{r,s,\boldsymbol{\gamma},\mathbf{k}} = \frac{\zeta_{r,\mathbf{k}}}{\zeta_{s,\mathbf{k}}}\,\alpha_{\mathbf{k},\boldsymbol{\gamma}},$$

$$\zeta_{s,\mathbf{k}} = \sum_{0 \leq |\mathbf{m}| \leq s} \prod_{j=1}^{d} [\pi(k_j - 1)]^{2m_j},$$

$$\alpha_{\mathbf{k},\boldsymbol{\gamma}} = \left(\sum_{\mathfrak{u} \subseteq [d]} \gamma_{d,\mathfrak{u}}^{-1} \prod_{j \in \mathfrak{u}} [\pi(k_j - 1)]^2 \right)^{-1}.$$