Exponential decay of reconstruction error from binary measurements of sparse signals

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for g rapidly decreasing to zero when λ increases.

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- Power decay is optimal since

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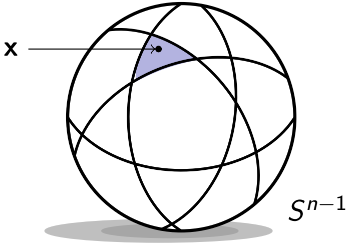

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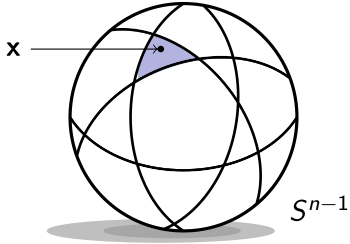

- Remedy: adaptive choice of dithers $\tau_1, \ldots, \tau_m$ in

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$$\mathbf{x}^{t+1} = \quad \mathbf{x}^t + \widehat{\mathbf{x} - \mathbf{x}^t}\ , \quad \text{so that} \quad \|\mathbf{x} - \mathbf{x}^{t+1}\|_2 \le R/4^{t+1}.$$

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- After T iterations, number of measurements is $m = qT$, and

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- Software step needed to compute the thresholds $\tau_i = \langle \mathbf{a}_i, \mathbf{x}^t \rangle$.

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- Cons: slow, post-quantization error not handled.

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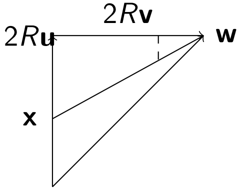

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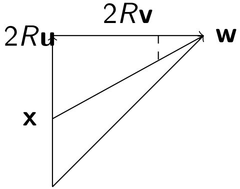

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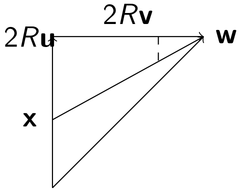

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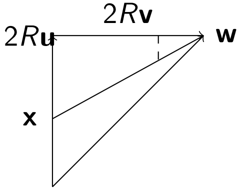

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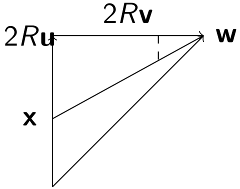

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- Pros: deterministic, fast, handles pre/post-quantization errors.

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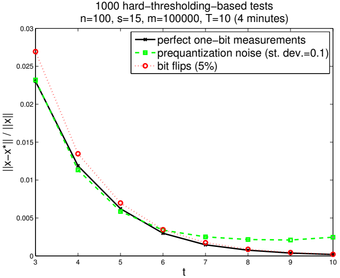
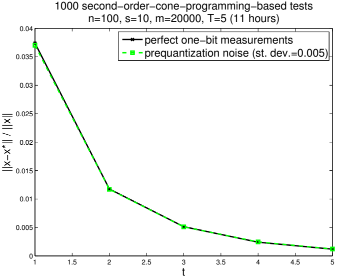
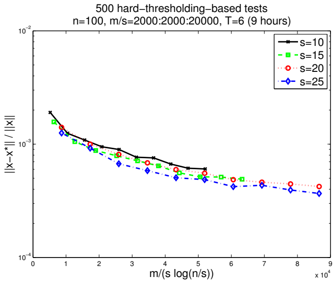
  for the hard-thresholding scheme.

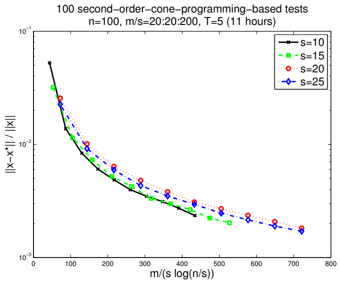
Numerical Illustration





Numerical Illustration, ctd





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- Sign Product Embedding Property: if $q \geq C\delta^{-6} s \ln(n/s)$, then with w/hp

$$\left| \frac{\sqrt{\pi/2}}{q} \langle \mathbf{A}\mathbf{w}, \mathrm{sign}(\mathbf{A}\mathbf{x}) \rangle - \langle \mathbf{w}, \mathbf{x} \rangle \right| \leq \delta$$

for all $\mathbf{w}, \mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{w}\|_0, \|\mathbf{x}\|_0 \leq s$ and $\|\mathbf{w}\|_2 = \|\mathbf{x}\|_2 = 1$.

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$$\mathbf{e} = \mathbf{A}\mathbf{u} \quad \text{with} \quad \begin{cases} \|\mathbf{u}\|_2 & \leq & d\|\mathbf{e}\|_2/\sqrt{q}, \\ \|\mathbf{u}\|_1 & \leq & d'\sqrt{s_*}\|\mathbf{e}\|_2/\sqrt{q}, \end{cases}$$

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- Restricted Isometry Property: if $q \geq C\delta^{-2} s \ln(n/s)$, then with w/hp

$$\left| \frac{1}{q}\|\mathbf{A}\mathbf{x}\|_2^2 - \|\mathbf{x}\|_2^2 \right| \leq \delta \|\mathbf{x}\|_2^2$$

  for all $\mathbf{x} \in \mathbb{R}^n$ with $\|\mathbf{x}\|_0 \leq s$.

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