

Nonlinear tensor product approximation

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Best multilinear approximation

We are interested in approximation of a multivariate function $f(x_1, \dots, x_d)$ by linear combinations of products $u^1(x_1) \cdots u^d(x_d)$ of univariate functions $u^i(x_i)$, $i = 1, \dots, d$.

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Definition

For a function $f(x_1, \dots, x_d)$ denote

$$\Theta_M(f)_X := \inf_{\{u_j^i\}_{j=1, \dots, M, i=1, \dots, d}} \|f(x_1, \dots, x_d) - \sum_{j=1}^M \prod_{i=1}^d u_j^i(x_i)\|_X$$

and for a function class F define

$$\Theta_M(F)_X := \sup_{f \in F} \Theta_M(f)_X.$$

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In the case $X = L_p$ we write p instead of L_p in the notation.

Multilinear dictionary

In other words we are interested in studying M -term approximations of functions with respect to the dictionary

$$\Pi^d := \{g(x_1, \dots, x_d) : g(x_1, \dots, x_d) = \prod_{i=1}^d u^i(x_i)\}$$

where $u^i(x_i)$ are arbitrary univariate functions. We discuss the case of 2π -periodic functions of d variables and approximate them in the L_p spaces. Denote by Π_p^d the normalized in L_p dictionary Π^d of 2π -periodic functions.

Mega problem

Problem

Find a simple algorithm such that for any $f \in L_p$ it provides after $M \leq m\phi(m)$ iterations an M -term with respect to Π^d approximant $A_M(f)$ such that

$$\|f - A_M(f)\|_p \leq C_1 \Theta_m(f)_p.$$

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Ideally, $\phi(m) = C_2$. Otherwise, the slower growing $\phi(m)$ the better.

What does give a hope?

- In the case $d = 2$, $p = 2$ the Pure Greedy Algorithm and the Orthogonal Greedy Algorithm (Orthogonal Matching Pursuit) solve the problem with $\phi(m) = 1$, $C_1 = 1$.

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- In the case $d = 2$, $p = 2$ the **Pure Greedy Algorithm** and the **Orthogonal Greedy Algorithm** (**Orthogonal Matching Pursuit**) solve the problem with $\phi(m) = 1$, $C_1 = 1$.
- For the trigonometric system we have.

Theorem (T., 2014)

Let \mathcal{D} be the normalized in L_p , $2 \leq p < \infty$, real d -variate trigonometric system. Then for any $f \in L_p$ the **Weak Chebyshev Greedy Algorithm** with weakness parameter t gives

$$\|f_{C(t,p,d)m \ln(m+1)}\|_p \leq C \sigma_m(f, \mathcal{D})_p. \quad (1)$$

Dictionary

Definition

A set of functions \mathcal{D} from a Banach space X is a **dictionary** if each $g \in \mathcal{D}$ has norm one ($\|g\| := \|g\|_X = 1$) and the closure of $\text{span}\mathcal{D}$ coincides with X .

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Definition

For a nonzero element $f \in X$ we denote by F_f a **norming (peak) functional** for f :

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The existence of such a functional is guaranteed by the Hahn-Banach theorem.

Chebyshev greedy algorithm

Weak Chebyshev Greedy Algorithm (WCGA)(t) Let $t \in (0, 1]$. For a given f_0 we inductively define for each $m \geq 1$

- $\varphi_m \in \mathcal{D}$ is any satisfying

$$|F_{f_{m-1}}(\varphi_m)| \geq t \sup_{g \in \mathcal{D}} |F_{f_{m-1}}(g)|.$$

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- Define

$$\Phi_m := \text{span}\{\varphi_j\}_{j=1}^m,$$

and define G_m to be the best approximant to f_0 from Φ_m .

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- Denote

$$f_m := f - G_m.$$

Best m -term approximation

Definition

We let $\Sigma_m(\mathcal{D})$ denote the collection of all functions (elements) in X which can be expressed as a linear combination of at most m elements of \mathcal{D} . Thus each function $f \in \Sigma_m(\mathcal{D})$ can be written in the form

$$f = \sum_{g \in \Lambda} c_g g, \quad \Lambda \subset \mathcal{D}, \quad \#\Lambda \leq m,$$

where the c_g are real or complex numbers.

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where the c_g are real or complex numbers.

Definition

For a function $f \in X$ we define its best m -term approximation error

$$\sigma_m(f) := \sigma_m(f, \mathcal{D}) := \inf_{a \in \Sigma_m(\mathcal{D})} \|f - a\|.$$

Nice bases

There are two fundamental systems: the d -variate trigonometric system $\mathcal{T}^d := \{e^{i(k,x)}\}$ and the prototype of the wavelets the \mathcal{U}^d system. We define the system $\mathcal{U} := \{U_I\}$ in the univariate case.

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$$U_n^+(x) := \sum_{k=0}^{2^n-1} e^{ikx} = \frac{e^{i2^n x} - 1}{e^{ix} - 1}, \quad n = 0, 1, 2, \dots;$$

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$$U_n^+(x) := \sum_{k=0}^{2^n-1} e^{ikx} = \frac{e^{i2^n x} - 1}{e^{ix} - 1}, \quad n = 0, 1, 2, \dots;$$

$$U_{n,k}^+(x) := e^{i2^n x} U_n^+(x - 2\pi k 2^{-n}), \quad k = 0, 1, \dots, 2^n - 1;$$

$$U_{n,k}^-(x) := e^{-i2^n x} U_n^+(-x + 2\pi k 2^{-n}), \quad k = 0, 1, \dots, 2^n - 1.$$

System \mathcal{U}

It will be more convenient for us to normalize in L_2 the system of functions $\{U_{m,k}^+, U_{n,k}^-\}$ and enumerate it by dyadic intervals. We write

$$U_I(x) := 2^{-n/2} U_{n,k}^+(x) \quad \text{with} \quad I = [(k + 1/2)2^{-n}, (k + 1)2^{-n});$$

$$U_I(x) := 2^{-n/2} U_{n,k}^-(x) \quad \text{with} \quad I = [k2^{-n}, (k + 1/2)2^{-n});$$

and

$$U_{[0,1)}(x) := 1.$$

System \mathcal{U}^d

In the multivariate case of $x = (x_1, \dots, x_d)$ we define the system \mathcal{U}^d as the tensor product of the univariate systems \mathcal{U} . Let $l = l_1 \times \dots \times l_d$, $l_j \in D$, $j = 1, \dots, d$, then

$$U_l(x) := \prod_{j=1}^d U_{l_j}(x_j).$$

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Both \mathcal{T}^d and \mathcal{U}^d have two fundamental features: orthogonality and tensor product structure.

Definition

We say that a dictionary \mathcal{D} has a tensor product structure if all its elements have a form of products $u^1(x_1) \cdots u^d(x_d)$ of univariate functions $u^i(x_i)$, $i = 1, \dots, d$.

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Any dictionary with tensor product structure is a subset of \mathcal{U}^d .

Function class W_q^r

Let

$$F_r(t) := 1 + 2 \sum_{k=1}^{\infty} k^{-r} \cos(kt - \pi r/2)$$

be the univariate Bernoulli kernel and let

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Definition

We define

$$W_q^r := \{f : f = F_r * \varphi, \quad \|\varphi\|_q \leq 1\},$$

where $*$ denotes the convolution.

Best m -term approximation with respect to \mathcal{U}^d

Theorem (T, 2000; Th1)

For $1 < q, p < \infty$ and big enough r we have

$$\sigma_m(W_q^r, \mathcal{U}^d)_p \asymp m^{-r} (\ln m)^{(d-1)r}.$$

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Theorem (T, 2000; Th2)

For any orthonormal system Ψ we have for $1 \leq q < \infty$

$$\sigma_m(W_q^r, \Psi)_2 \gg m^{-r} (\ln m)^{(d-1)r}.$$

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Thus, the \mathcal{U}^d is an ideal orthogonal system.

Bilinear approximation

In the case $d = 2$ the **multilinear approximation** problem is a classical problem of **bilinear approximation**. There are known results on the rate of decay of errors of best bilinear approximation in L_p under different smoothness assumptions on f . We only mention some known results for classes of functions W_q^r . The problem of estimating $\Theta_M(f)_2$ in case $d = 2$ (best M -term bilinear approximation in L_2) is a classical one and was considered for the first time by **E. Schmidt in 1907**. For many function classes F an asymptotic behavior of $\Theta_M(F)_p$ is known. For instance

Theorem (T, 1986)

In the case $d = 2$ for $r > 1$ and $1 \leq q \leq p \leq \infty$ we have

$$\Theta_M(W_q^r)_p \asymp M^{-2r+(1/q-\max(1/2,1/p))_+} \quad (2)$$

Multilinear approximation

In the case $d > 2$ almost nothing is known. There is an upper estimate in the case $q = p = 2$

Theorem (T, 1988)

For $r > 0$ we have

$$\Theta_M(W_2^r)_2 \ll M^{-rd/(d-1)}. \quad (3)$$

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The above theorems show that the rate $M^{-r}(\log M)^{(d-1)r}$ of best M -term approximation with respect to the basis \mathcal{U}^d , which has a tensor structure, is not as good as best M -term approximation with respect to Π^d (we have exponent r for \mathcal{U}^d instead of $\frac{rd}{d-1}$ for Π^d).

Known lower bounds

New results of this talk are around the bound (3). First of all we discuss the lower bound matching the upper bound (3). In the case $d = 2$ the lower bound

$$\Theta_M(W_p^r)_p \gg M^{-2r}, \quad 1 \leq p \leq \infty. \quad (4)$$

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A stronger result

$$\Theta_M(W_\infty^r)_1 \gg M^{-2r} \quad (5)$$

follows from Theorem 1.1 in [T, 1992].

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Theorem (Bazarkhanov and T., 2014)

$$\Theta_M^b(W_\infty^r)_1 \gg (M \ln M)^{-\frac{rd}{d-1}}.$$

Upper bounds

Secondly, we discuss some upper bounds which extend the bound (3). The relation (2) shows that for $2 \leq p \leq \infty$ in the case $d = 2$ one has

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Theorem (Bazarkhanov and T., 2014)

Let $2 \leq p < \infty$ and $r > (d - 1)/d$. Then

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$$\Theta_M(W_2^r)_p \ll \left(\frac{M}{(\log M)^{d-1}} \right)^{-\frac{rd}{d-1}}.$$

The proof of the above theorem is not constructive. It goes by induction and uses a nonconstructive bound in the case $d = 2$.

TGA

We define a well-known **Thresholding Greedy Algorithm** with respect to a basis. It is convenient for us to enumerate the basis functions by dyadic intervals. Assume a given system Ψ of functions ψ_I indexed by dyadic intervals can be enumerated in such a way that $\{\psi_{I_j}\}_{j=1}^{\infty}$ is a basis for L_p . Then we define the greedy algorithm $G^p(\cdot, \Psi)$ as follows. Let

$$f = \sum_{j=1}^{\infty} c_{I_j}(f, \Psi) \psi_{I_j}, \quad c_I(f, p, \Psi) := \|c_I(f, \Psi) \psi_I\|_p.$$

Then $c_I(f, p, \Psi) \rightarrow 0$ as $|I| \rightarrow 0$.

TGA continue

Denote Λ_m a set of m dyadic intervals I such that

$$\min_{I \in \Lambda_m} c_I(f, p, \Psi) \geq \max_{J \notin \Lambda_m} c_J(f, p, \Psi).$$

We define $G^P(\cdot, \Psi)$ by formula

$$G_m^P(f, \Psi) := \sum_{I \in \Lambda_m} c_I(f, \Psi) \psi_I.$$

TGA is good

It is proved in [T., 2000] that for $1 < q, p < \infty$ and big enough r

$$\begin{aligned} \sup_{f \in W_q^r} \|f - G_M^p(f, \mathcal{U}^d)\|_p &\asymp \sigma_M(W_q^r, \mathcal{U}^d)_p \\ &\asymp M^{-r} (\log M)^{(d-1)r}. \end{aligned} \tag{7}$$

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The above relation (7) illustrates two phenomena: (I) for the class W_q^r the simple Thresholding Greedy Algorithm provides near best M -term approximation; (II) the rate $M^{-r}(\log M)^{(d-1)r}$ of best M -term approximation with respect to the basis \mathcal{U}^d , which has a tensor structure, is not as good as best M -term approximation with respect to Π^d (we have exponent r for \mathcal{U}^d instead of $\frac{rd}{d-1}$ for Π^d).

Constructive multilinear approximation

We use two very different greedy-type algorithms to provide a constructive multilinear approximant. Surprisingly, these two algorithms gave the same error bound.

Theorem (Bazarkhanov and T., 2014)

For big enough r the following constructive upper bound for $2 \leq p < \infty$ holds

$$\Theta_M(W_2^r)_p \ll \left(\frac{M}{(\ln M)^{d-1}} \right)^{-\frac{rd}{d-1} + \frac{\beta}{d-1}}.$$

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This constructive upper bound has an extra term $\frac{\beta}{d-1}$ in the exponent compared to the best M -term approximation. It would be interesting to find a constructive way to obtain the near best approximation in this case.

Trigonometric system. Some history

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Key: constructive, uses Gaussian sums to prove an inequality for trigonometric polynomials

$$\sigma_m(f, \mathcal{T})_\infty \leq CN^{3/2}m^{-1}\|f\|_1, \quad f \in \mathcal{T}(N). \quad (T1)$$

More history

It is easy to see that (T1) follows from

$$\sigma_m(f, \mathcal{T})_\infty \leq CN^{1/2}m^{-1}\|f\|_A, \quad f \in \mathcal{T}(N). \quad (T2)$$

where

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Devore and T. (1995)

$$\sigma_m(f, \mathcal{T})_\infty \leq Cm^{-1/2}(\ln(1 + N/m))^{1/2}\|f\|_A, \quad f \in \mathcal{T}(N). \quad (T3)$$

In a certain sense (T3) is much stronger than (T2). However, the proof was nonconstructive.

History continues

Dilworth, Kutzarova, and T. (2002)

$$\sigma_m(f, \mathcal{T})_p \leq C(p)m^{-1/2}\|f\|_A, \quad 2 \leq p < \infty, \quad f \in \mathcal{T}(N). \quad (T4)$$

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Dilworth, Kutzarova, and T. (2002)

$$\sigma_m(f, \mathcal{T})_p \leq C(p)m^{-1/2}\|f\|_A, \quad 2 \leq p < \infty, \quad f \in \mathcal{T}(N). \quad (T4)$$

T. (2005) gave a constructive proof of (T3). This proof and the proof of (T4) use the Weak Chebyshev Greedy Algorithm.

History continues

Dilworth, Kutzarova, and T. (2002)

$$\sigma_m(f, \mathcal{T})_p \leq C(p)m^{-1/2}\|f\|_A, \quad 2 \leq p < \infty, \quad f \in \mathcal{T}(N). \quad (T4)$$

T. (2005) gave a constructive proof of (T3). This proof and the proof of (T4) use the Weak Chebyshev Greedy Algorithm. In particular, (T4) implies

$$\sigma_m(f, \mathcal{T})_p \leq C(p)N^{1/2}m^{-1/2}\|f\|_2, \quad 2 \leq p < \infty, \quad f \in \mathcal{T}(N). \quad (T5)$$

Best multilinear approximation

Lemma (Bazarkhanov and T., 2014)

Let $f \in T(\mathbf{N})$. Denote $v(\mathbf{N}) := \prod_{j=1}^d \bar{N}_j$. Then for $2 \leq p < \infty$ one has

$$\Theta_M(f)_p \ll v(\mathbf{N})^{1-\frac{1}{p}} (\bar{M})^{-1} \|f\|_2, \quad \bar{M} = \max(M, 1).$$

TGA-type algorithm

Lemma (Bazarkhanov and T., 2014)

Suppose that $f \in T(\mathbf{N})$. Denote $v(\mathbf{N}) := \prod_{j=1}^d \bar{N}_j$. Then for $1 \leq q \leq p \leq \infty$

$$\Theta_m(f)_p \ll v(\mathbf{N})^\beta (\bar{m})^{-\beta} \|f\|_q, \quad \beta := \frac{1}{q} - \frac{1}{p}, \quad \bar{m} := \max(1, m). \quad (8)$$

The bound (8) is realized by a simple greedy-type algorithm.

General greedy-type algorithms

Lemma (Bazarkhanov and T., 2014)

Let $f \in T(\mathbf{N})$. Then for $2 \leq p < \infty$

$$\Theta_m(f)_p \ll v(\mathbf{N})^{\frac{1}{2} - \frac{1}{pd}} (\bar{m})^{-1/2} \|f\|_2.$$

The above bound is realized by the *Weak Chebyshev Greedy Algorithm* with weakness parameter t .