

High-Order Quasi Monte-Carlo Integration  
for  
Bayesian Inversion of Parametric Operator Equations

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## Outline

- Infinite-Dimensional Parametric Operator Equations
- Quasi Monte-Carlo Integration and High Order Digital Nets
- Quasi Monte-Carlo Petrov-Galerkin FEM convergence rate
- Bayesian Inverse Problem
- Numerical Results
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# Infinite-Dimensional Parametric Operator Equations

- Example: Linear, Affine-Parametric Operator Equation

$$\text{Given } f \in \mathcal{Y}' , \text{ for every } \mathbf{y} \in U \text{ find } u(\mathbf{y}) \in \mathcal{X} : \quad A(\mathbf{y}) u(\mathbf{y}) = f . \quad (1)$$

Here

$$A(\mathbf{y}) = A_0 + \sum_{j \geq 1} y_j A_j \in \mathcal{L}(\mathcal{X}; \mathcal{Y}') , \quad \forall \mathbf{y} := (y_j)_{j \geq 1} \in U := [-1/2, 1/2]^{\mathbb{N}} . \quad (2)$$

- Assumptions:

$$\text{Small Fluctuations: } \sum_{j \geq 1} \|A_j\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} \text{ "small" , } \quad \text{Sparsity: } \exists 0 < p < 1 : \quad \sum_{j \geq 1} \|A_j\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')}^p < \infty . \quad (3)$$

- Example: Karhúnen-Loeve expansion:  $a(x, \mathbf{y}) = \bar{a}(x) + \sum_{j \geq 1} y_j \psi_j(x)$

$$A(\mathbf{y}) = -\nabla_x \cdot a(x, \mathbf{y}) \nabla_x = \underbrace{-\nabla_x \cdot \bar{a}(x) \nabla_x}_{A_0} + \sum_{j \geq 1} \underbrace{-\nabla_x \cdot \psi_j(x) \nabla_x}_{A_j} , \quad \mathcal{X} = \mathcal{Y} = H_0^1(D) .$$

- Goal: given  $G \in \mathcal{X}'$ , find  $\mathbb{E}[G(u(\cdot))]$  with respect to  $\mathbf{y} \in U$ , i.e.,

$$I(G(u)) := \int_U G(u(\mathbf{y})) d\mathbf{y} . \quad (4)$$

# Infinite-Dimensional Parametric Operator Equations

- Strategy:

1. **Dimension Truncation:** truncate (2) to  $s$  terms,
2. solve  $s$ -dimensional equation (1) by **Petrov-Galerkin discretization** from  $\{\mathcal{X}^h\} \subset \mathcal{X}$ ,
3. approximate  $s$ -dimensional integral using **QMC integration**,

$$\frac{1}{N} \sum_{n=0}^{N-1} G(u_s^h(\mathbf{y}_n - \frac{1}{2})) , \quad (5)$$

where  $\mathbf{y}_0, \dots, \mathbf{y}_{N-1} \in [0, 1]^s$  denote  $N$  points from a higher order digital net.

4. Error bounds explicit w.r. to  $N$ ,  $h$  and truncation dimension  $s$ .

## Infinite-Dimensional Parametric Operator Equations

- Variational Formulation:  $\mathbf{a}_j(\cdot, \cdot) : \mathcal{X} \times \mathcal{Y} \rightarrow \mathbb{R}$  via

$$\forall v \in \mathcal{X}, w \in \mathcal{Y} : \quad \mathbf{a}_j(v, w) = {}_{\mathcal{Y}}\langle w, A_j v \rangle_{\mathcal{Y}'}, \quad j = 0, 1, 2, \dots .$$

### Assumption:

The sequence  $\{A_j\}_{j \geq 0}$  satisfies:

1. the *nominal operator*  $A_0 \in \mathcal{L}(\mathcal{X}, \mathcal{Y}')$  is boundedly invertible:

$$\inf_{0 \neq v \in \mathcal{X}} \sup_{0 \neq w \in \mathcal{Y}} \frac{\mathbf{a}_0(v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu_0 > 0, \quad \inf_{0 \neq w \in \mathcal{Y}} \sup_{0 \neq v \in \mathcal{X}} \frac{\mathbf{a}_0(v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu_0 > 0. \quad (6)$$

2. the *fluctuation operators*  $\{A_j\}_{j \geq 1}$  are small with respect to  $A_0$ : exists  $0 < \kappa < 2$  such that

$$\sum_{j \geq 1} \beta_j \leq \kappa < 2, \quad \text{where} \quad \beta_j := \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X}, \mathcal{Y}')} , \quad j = 1, 2, \dots . \quad (7)$$

## Infinite-Dimensional Parametric Operator Equations

Assumption is sufficient for bounded invertibility of  $A(\mathbf{y})$  uniformly w.r. to  $\mathbf{y} \in U$ :  
for every realization  $\mathbf{y} \in U$  of the parameter vector

$$\mathbf{a}(\mathbf{y}; v, w) := {}_{\mathcal{Y}}\langle w, A(\mathbf{y})v \rangle_{\mathcal{Y}'}, \quad (8)$$

satisfies uniform (with respect to  $\mathbf{y} \in U$ ) inf-sup conditions: with  $\mu = (1 - \kappa/2)\mu_0 > 0$ ,

$$\forall \mathbf{y} \in U : \quad \inf_{0 \neq v \in \mathcal{X}} \sup_{0 \neq w \in \mathcal{Y}} \frac{\mathbf{a}(\mathbf{y}; v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu, \quad \inf_{0 \neq w \in \mathcal{Y}} \sup_{0 \neq v \in \mathcal{X}} \frac{\mathbf{a}(\mathbf{y}; v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu. \quad (9)$$

For every  $f \in \mathcal{Y}'$  and for every  $\mathbf{y} \in U$ , the parametric problem

$$\text{find } u(\mathbf{y}) \in \mathcal{X} : \quad \mathbf{a}(\mathbf{y}; u(\mathbf{y}), w) = {}_{\mathcal{Y}}\langle w, f \rangle_{\mathcal{Y}'} \quad \forall w \in \mathcal{Y} \quad (10)$$

admits unique solution  $u(\mathbf{y})$  which satisfies the a-priori estimate

$$\|u(\mathbf{y})\|_{\mathcal{X}} \leq \frac{1}{\mu} \|f\|_{\mathcal{Y}'}. \quad (11)$$

## Infinite-Dimensional Parametric Operator Equations

- Parametric regularity of solutions

$$\forall \mathbf{y} \in U : \quad \|(\partial_{\mathbf{y}}^{\boldsymbol{\nu}} u)(\mathbf{y})\|_{\mathcal{X}} \leq C_0 |\boldsymbol{\nu}|! \boldsymbol{\beta}^{\boldsymbol{\nu}} \|f\|_{\mathcal{Y}'} \quad \text{for all } \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \text{ with } |\boldsymbol{\nu}| < \infty, \quad (12)$$

where  $0! := 1$ ,  $\boldsymbol{\beta}^{\boldsymbol{\nu}} := \prod_{j \geq 1} \beta_j^{\nu_j}$ , with  $\beta_j = \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}$  and  $|\boldsymbol{\nu}| = \sum_{j \geq 1} \nu_j$ .

- Spatial regularity: *scales of smoothness spaces*  $\{\mathcal{X}_t\}_{t \geq 0}$ ,  $\{\mathcal{Y}_t\}_{t \geq 0}$ , with

$$\begin{aligned} \mathcal{X} &= \mathcal{X}_0 \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset \dots, & \mathcal{Y} &= \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \dots, & \text{and} \\ \mathcal{X}' &= \mathcal{X}'_0 \supset \mathcal{X}'_1 \supset \mathcal{X}'_2 \supset \dots, & \mathcal{Y}' &= \mathcal{Y}'_0 \supset \mathcal{Y}'_1 \supset \mathcal{Y}'_2 \supset \dots. \end{aligned} \quad (13)$$

- Uniform regularity shift (sufficient for single-level QMC-PG): for  $0 < t \leq \bar{t}$ ,

$$\forall \mathbf{y} \in U : \quad f \in \mathcal{Y}'_t \implies u(\mathbf{y}) = A(\mathbf{y})^{-1} f \in \mathcal{X}_t.$$

- Mixed regularity shift (necessary for multi-level QMC-PG):

$$f \in \mathcal{Y}'_t \implies \sup_{\mathbf{y} \in U} \|(\partial_{\mathbf{y}}^{\boldsymbol{\nu}} u)(\mathbf{y})\|_{\mathcal{X}_t} \leq C_0 |\boldsymbol{\nu}|! \boldsymbol{\beta}_t^{\boldsymbol{\nu}} \|f\|_{\mathcal{Y}'_t} \quad \text{for all } \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \text{ with } |\boldsymbol{\nu}| < \infty.$$

## Infinite-Dimensional Parametric Operator Equations

- Best  $N$ -term polynomial chaos approximation

Under Assumption,  $u(\mathbf{y}) : U \rightarrow \mathcal{X}$  can be expanded in (unconditionally convergent in  $L^2(U; d\mathbf{y})$ ) gpc series

$$\forall \mathbf{y} \in U : \quad u(\mathbf{y}) = \sum_{\mathcal{F}} u_{\nu} L_{\nu}(\mathbf{y}) , \quad \text{where } u_{\nu} = (u, L_{\nu})_{L^2(U; d\mathbf{y})} \in \mathcal{X} .$$

Here  $\mathcal{F} = \{\nu \in \mathbb{N}_0^{\mathbb{N}} : |\nu| < \infty\}$ , and  $L_{\nu}$  is the ( $L^2(U; d\mathbf{y})$ -normalized) tensorized Legendre polynomial

$$\forall \nu \in \mathcal{F}, \forall \mathbf{y} \in U : \quad L_{\nu}(\mathbf{y}) := \prod_{j \geq 1} L_{\nu_j}(y_j) \quad (\text{note } L_0 \equiv 1) .$$

- [Cohen & DeVore & CS (2011), (Chkifa & Cohen & CS 2014)]

Assume that  $\beta \in \ell^p(\mathbb{N})$  for some  $0 < p < 1$ . Then for every  $N \in \mathbb{N}$  exists  $\Lambda \subset \mathcal{F}$  such that, for  $q = 2, \infty$

$$\#(\Lambda) = N \quad \text{and} \quad \|u - u_{\Lambda}\|_{L^q(U, d\mathbf{y}; \mathcal{X})} \leq C(q) N^{-(1/p-1/q')} .$$

$q'$  conjugate of  $q = 2, \infty$ , constant  $C > 0$  independent of  $N$  and of dimension.

- Proof nonconstructive. “Constructive versions” (Chkifa & Cohen & DeVore & CS 2012-2014).



## Infinite-Dimensional Parametric Operator Equations

- *Petrov-Galerkin discretization:*

Let  $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$  and  $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$  dense families of subspaces in  $\mathcal{X}$  and  $\mathcal{Y}$ .

- *Approximation Properties:* for  $0 < t \leq \bar{t}$  and  $0 < t' \leq \bar{t}'$ , and for  $0 < h \leq h_0$ , there hold

$$\begin{aligned} \forall v \in \mathcal{X}_t : \quad & \inf_{v^h \in \mathcal{X}^h} \|v - v^h\|_{\mathcal{X}} \leq C_t h^t \|v\|_{\mathcal{X}_t} , \\ \forall w \in \mathcal{Y}_{t'} : \quad & \inf_{w^h \in \mathcal{Y}^h} \|w - w^h\|_{\mathcal{Y}} \leq C_{t'} h^{t'} \|w\|_{\mathcal{Y}_{t'}} . \end{aligned} \tag{14}$$

$$\forall 0 \leq t \leq \bar{t} : \quad \sup_{\mathbf{y} \in U} \|A(\mathbf{y})^{-1}\|_{\mathcal{L}(\mathcal{Y}_{t'}, \mathcal{X}_t)} < \infty . \tag{15}$$

- *Stability:* assume that  $(\mathcal{X}^h, \mathcal{Y}^h)$  satisfy discr. inf-sup condition for  $A_0$ .

Then there hold uniform (with respect to  $\mathbf{y} \in U$ ) discrete inf-sup conditions

$$\forall \mathbf{y} \in U : \quad \inf_{0 \neq v^h \in \mathcal{X}^h} \sup_{0 \neq w^h \in \mathcal{Y}^h} \frac{\mathbf{a}(\mathbf{y}; v^h, w^h)}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 , \tag{16}$$

$$\forall \mathbf{y} \in U : \quad \inf_{0 \neq w^h \in \mathcal{Y}^h} \sup_{0 \neq v^h \in \mathcal{X}^h} \frac{\mathbf{a}(\mathbf{y}; v^h, w^h)}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 . \tag{17}$$

## Infinite-Dimensional Parametric Operator Equations

- For every  $0 < h \leq h_0$  and for every  $\mathbf{y} \in U$ , Petrov-Galerkin approximation

$$\text{find } u^h(\mathbf{y}) \in \mathcal{X}^h : \quad \mathbf{a}(\mathbf{y}; u^h(\mathbf{y}), w^h) = {}_{\mathcal{Y}}\langle w^h, f \rangle_{\mathcal{Y}'} \quad \forall w^h \in \mathcal{Y}^h, \quad (18)$$

admits a unique solution  $u^h(\mathbf{y})$  which satisfies the a-priori estimate

$$\|u^h(\mathbf{y})\|_{\mathcal{X}} \leq \frac{1}{\bar{\mu}} \|f\|_{\mathcal{Y}'}. \quad (19)$$

- Quasioptimality: exists a constant  $C > 0$  such that for all  $\mathbf{y} \in U$

$$\|u(\mathbf{y}) - u^h(\mathbf{y})\|_{\mathcal{X}} \leq \frac{C}{\bar{\mu}} \inf_{0 \neq v^h \in \mathcal{X}^h} \|u(\mathbf{y}) - v^h\|_{\mathcal{X}}. \quad (20)$$

- *Convergence Rate*: Ex.  $C > 0$  such that for every  $f \in \mathcal{Y}'_t$  with  $0 < t \leq \bar{t}$  as  $h \rightarrow 0$

$$\|u(\mathbf{y}) - u^h(\mathbf{y})\|_{\mathcal{X}} \leq C h^t \|f\|_{\mathcal{Y}'_t}. \quad (21)$$

- Dimension-Truncation: For  $\mathbf{y} \in U$  and  $s \in \mathbb{N}$ ,  $(y_1, y_2, \dots, y_s, 0, 0, \dots) \in U$ , so

**all bounds valid for  $[-1/2, 1/2]^s$  uniformly w.r. to  $s$ .**

## Infinite-Dimensional Parametric Operator Equations

- *Regularity:*

$$\exists 0 < t' \leq \bar{t} : G(\cdot) \in \mathcal{X}'_{t'} , \quad (22)$$

- *Adjoint Regularity:* exists  $C_{t'} > 0$  such that for every  $\mathbf{y} \in U$ ,

$$\forall \mathbf{y} \in U : w(\mathbf{y}) = (A^*(\mathbf{y}))^{-1}G \in \mathcal{Y}'_{t'} , \quad \|w(\mathbf{y})\|_{\mathcal{Y}'_{t'}} \leq C_{t'} \|G\|_{\mathcal{X}'_{t'}} . \quad (23)$$

- *Superconvergence (Aubin-Nitsche):*

for every  $f \in \mathcal{Y}'_t$  with  $0 < t \leq \bar{t}$ , for every  $G(\cdot) \in \mathcal{X}'_{t'}$  with  $0 < t' \leq \bar{t}$

$$\sup_{\mathbf{y} \in U} |G(u(\mathbf{y})) - G(u^h(\mathbf{y}))| \leq C h^\tau \|f\|_{\mathcal{Y}'_t} \|G\|_{\mathcal{X}'_{t'}} . \quad (24)$$

where  $0 < \tau := t + t' \leq 2\bar{t}$ .

## Dimension Truncation

Assume the  $A_j$  are enumerated s.t.  $\beta_j := \|A_0^{-1}A_j\|_{\mathcal{X}}$  satisfy

$$\beta_1 \geq \beta_2 \geq \dots \geq \beta_j \geq \dots . \quad (25)$$

Then, for every  $f \in \mathcal{Y}'$ , every  $\mathbf{y} \in U$  and for every  $s \in \mathbb{N}$ ,

$$\sup_{\mathbf{y} \in U} \|u(\mathbf{y}) - u_s(\mathbf{y})\|_{\mathcal{X}} \leq \frac{C}{\mu} \|f\|_{\mathcal{Y}'} \sum_{j \geq s+1} \beta_j . \quad (26)$$

For every  $G(\cdot) \in \mathcal{X}'$ ,

$$|I(G(u)) - I(G(u_s))| \leq \frac{\tilde{C}}{\mu} \|f\|_{\mathcal{Y}'} \|G\|_{\mathcal{X}'} \left( \sum_{j \geq s+1} \beta_j \right)^2 \quad (27)$$

for  $\tilde{C} > 0$  independent of  $s$ ,  $f$  and  $G$ .

If conditions (25) hold, for any  $0 < p < 1$  and for any  $s \in \mathbb{N}$  holds the dimension-truncation error bound

$$\sum_{j \geq s+1} \beta_j \leq \min \left( \frac{1}{1/p - 1}, 1 \right) \left( \sum_{j \geq 1} \beta_j^p \right)^{1/p} s^{-(1/p-1)} .$$

## Quasi Monte-Carlo Integration and High Order Digital Nets

- Consider *general*  $s$ -variate integrand  $F \in C^0([0, 1]^s)$ . Approximate  $s$ -dimensional integral

$$I_s(F) := \int_{[0,1]^s} F(\mathbf{y}) \, d\mathbf{y} \quad (28)$$

where  $F(\mathbf{y}) = G(u_s^h(\mathbf{y} - \frac{1}{2}))$  by

- $N$ -point QMC method: an equal-weight quadrature rule

$$Q_{N,s}(F) := \frac{1}{N} \sum_{n=0}^{N-1} F(\mathbf{y}_n) , \quad (29)$$

with judiciously chosen points  $\mathbf{y}_0, \dots, \mathbf{y}_{N-1} \in [0, 1]^s$ .

## Quasi Monte-Carlo Integration and High Order Digital Nets

**Theorem 1** [Dick, Gia, Kuo, Nuyens, CS 2013]

Let  $s \geq 1$  and  $N = b^m$  for  $m \geq 1$  and prime  $b$ . Let  $\boldsymbol{\beta} = (\beta_j)_{j \geq 1}$  be a sequence of positive numbers s.t.

$$\exists 0 < p < 1 : \sum_{j=1}^{\infty} \beta_j^p < \infty . \quad (30)$$

Define  $\boldsymbol{\beta}_{\{1:s\}} = (\beta_j)_{1 \leq j \leq s}$  and

$$\alpha := \lfloor 1/p \rfloor + 1 . \quad (31)$$

Assume for every  $s \in \mathbb{N}$ , for every  $\mathbf{y} \in [-1/2, 1/2]^{\mathbb{N}}$

$$\forall \boldsymbol{\nu} \in \{0, 1, \dots, \alpha\}^s : |(\partial_{\mathbf{y}}^{\boldsymbol{\nu}} F)(\mathbf{y}_{\{1:s\}})| \leq c |\boldsymbol{\nu}|! \boldsymbol{\beta}_{\{1:s\}}^{\boldsymbol{\nu}} \quad (32)$$

Then, an interlaced polynomial lattice rule of order  $\alpha \geq 1$  with  $N$  points can be constructed using a fast component-by-component algorithm, with **cost**  $\mathcal{O}(\alpha s N \log N + \alpha^2 s^2 N)$  **operations**, such that

$$|I_s(F) - Q_{N,s}(F)| \leq C_{\alpha, \boldsymbol{\beta}, b, p} N^{-1/p} ,$$

where  $C_{\alpha, \boldsymbol{\beta}, b, p} < \infty$  is independent of  $s$  and  $N$ .

## Combined Error Bound

**Theorem 2** [Single-Level High-Order Quasi Monte-Carlo Galerkin Error bound]

1. Approximate  $I(G(u(\cdot)))$  by dimension-truncation and interlaced polynomial lattice rule (5)

order  $\alpha = \lfloor 1/p \rfloor + 1$ , with  $N = b^m$  points in  $s$  dimensions,

with Petrov-Galerkin discretization in  $D$  with subspace  $\mathcal{X}^h$  with  $M_h = \dim(\mathcal{X}^h)$  DoF, cost  $\mathcal{O}(M_h)$ .

Then ex.  $C > 0$  independent of  $s$ ,  $h$  and  $N$  such that with  $\tau = t + t'$

$$|I(G(u)) - Q_{N,s}(G(u_s^h))| \leq C \left( (s^{-2(1/p-1)} + N^{-1/p}) \|f\|_{\mathcal{Y}'} \|G(\cdot)\|_{\mathcal{X}'} + h^\tau \|f\|_{\mathcal{Y}_t'} \|G(\cdot)\|_{\mathcal{X}_t'} \right).$$

2. Cost for evaluation of  $Q_{N,s}(G(u_s^h))$  is  $\mathcal{O}(sNM_h)$  operations.
3. Cost for CBC construction of the interlaced polynomial lattice rule

$$\mathcal{O}(\alpha s N \log N + \alpha^2 s^2 N) \text{ operations,} \quad \mathcal{O}(\alpha s N) \text{ memory .}$$

**Proof**

$$I(G(u)) - Q_{N,s}(G(u_s^h)) = [I(G(u)) - I(G(u_s))] + [I(G(u_s)) - I(G(u_s^h))] + [I(G(u_s^h)) - Q_{N,s}(G(u_s^h))] .$$

□

## Bayesian Inverse Problems

- uncertain input data  $u(\mathbf{y}) = \langle u \rangle + \sum_{j \geq 1} y_j \psi_j \in X$ , uniform prior  $\pi_0$  on  $u$ ,
- forward equation:  $A(u; q) = f$ ,  $f \in \mathcal{Y}'$  known, forward solution map  $G(u(\mathbf{y}))$  holomorphic w.r. to  $\mathbf{y}$ .
- noisy observation data:  $\delta = \mathcal{G}(u) + \eta \in Y$ ,  $\mathcal{G} = \mathcal{O} \circ G : X \rightarrow Y$ ,  $\eta \sim N(0, \Gamma) \in Y = \mathbb{R}^K$ ,
- Qol:  $\phi(u)$ .

**Bayes' Estimate:**

$$I_\Gamma(\phi(u)) := \frac{1}{Z} \int_{\mathbf{y} \in [-1/2, 1/2]^{\mathbb{N}}} \underbrace{\exp\left(-\frac{1}{\Gamma} \|\delta - \mathcal{G}(u(\mathbf{y}))\|_Y^2\right) \phi(u(\mathbf{y})) \pi_0(d\mathbf{y})}_{=: F(\mathbf{y})}$$

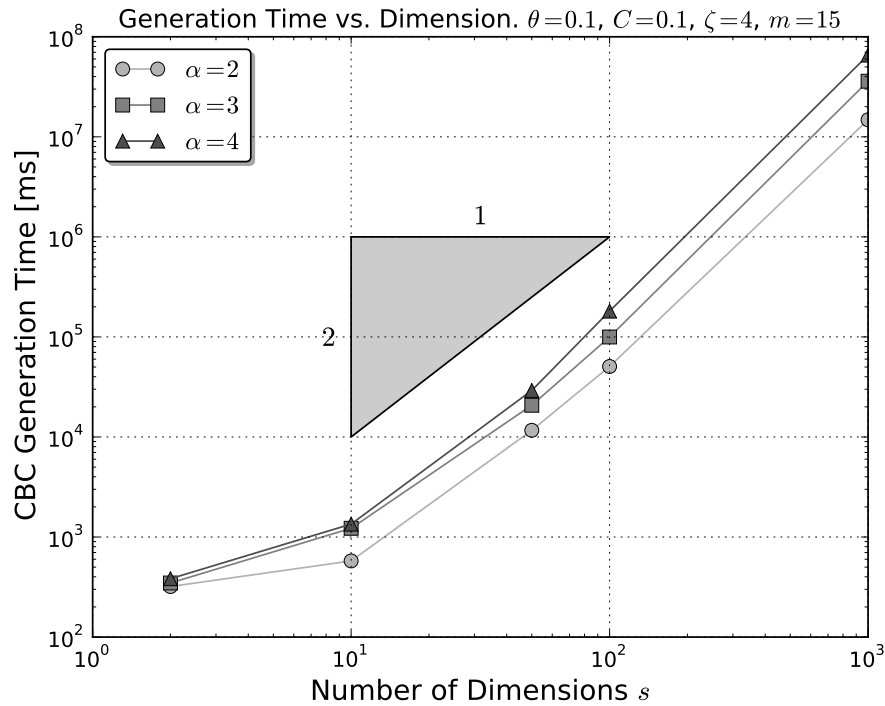
**Theorem 3** [High-Order Quasi Monte-Carlo for Bayesian Inverse Problems]

For every  $\Gamma > 0$ , the parametric, deterministic integrand function  $F(\mathbf{y})$  satisfies the derivative bounds (32).

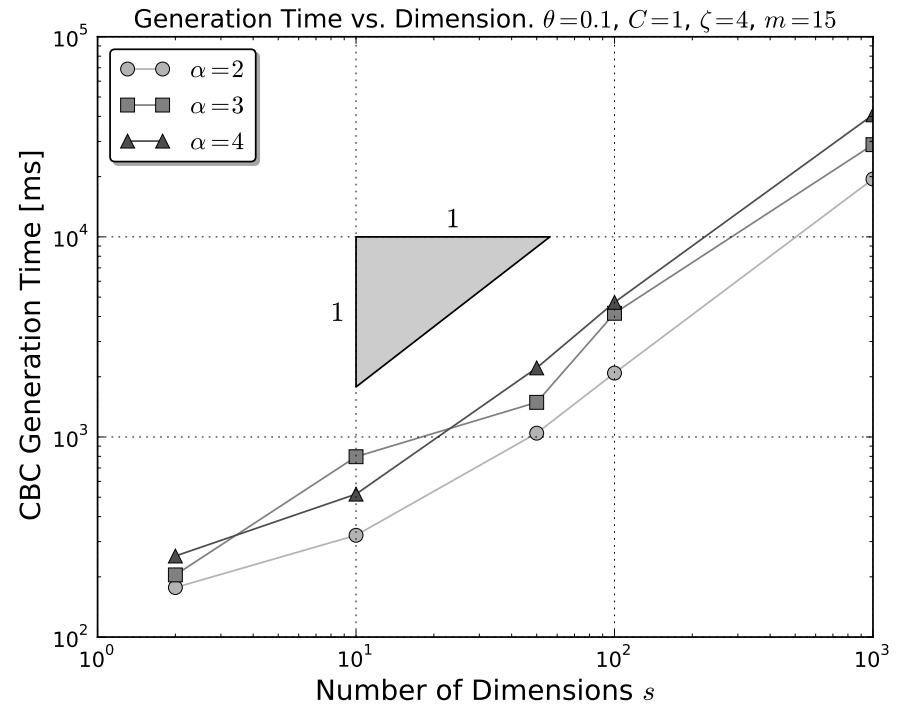
The preceding error bounds apply to Bayesian Estimation.

**Proof:** Analytic continuation and Cauchy's Theorem. □



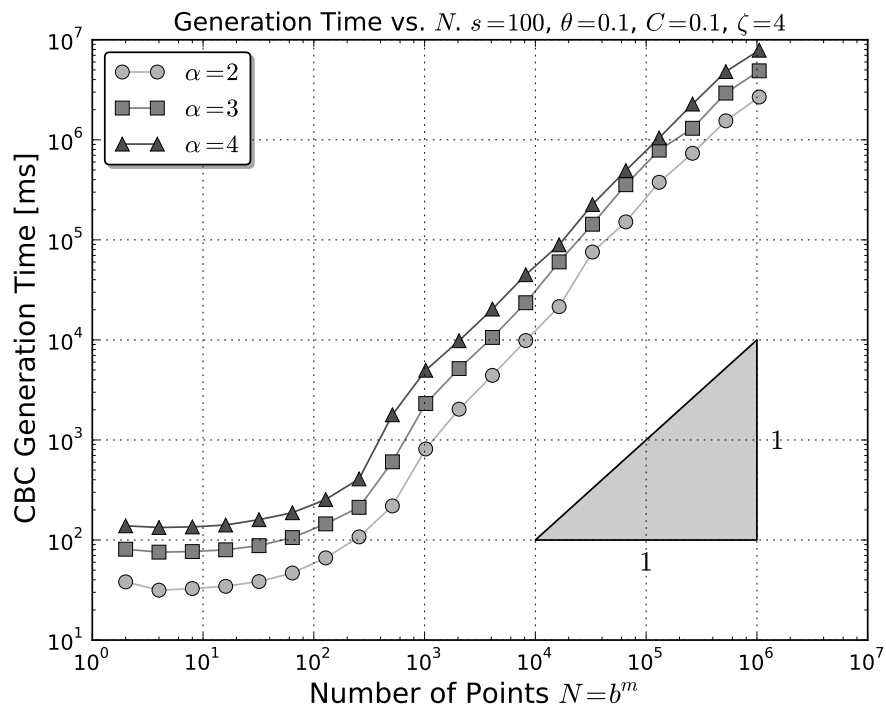


(a) SPOD,  $C = 0.1$

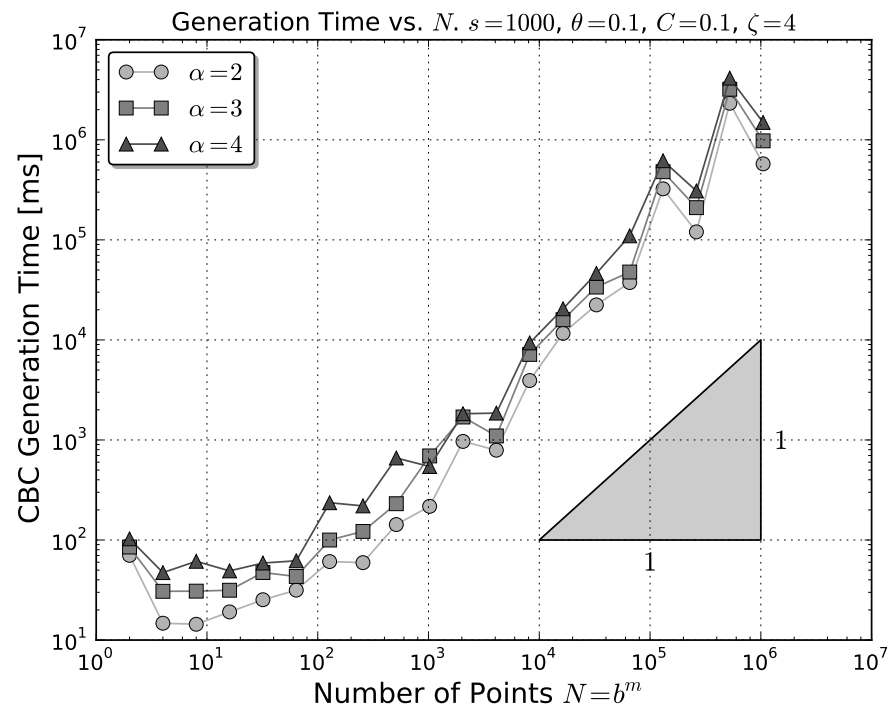


(b) product,  $C = 1$

Figure 1: CPU time required for the construction of generating vectors of varying order  $\alpha = 2, 3, 4$  for product and SPOD weights vs. the dimension  $s$  in (a) and (b) and vs. the number of points  $N = 2^m$  in (a) and (b).

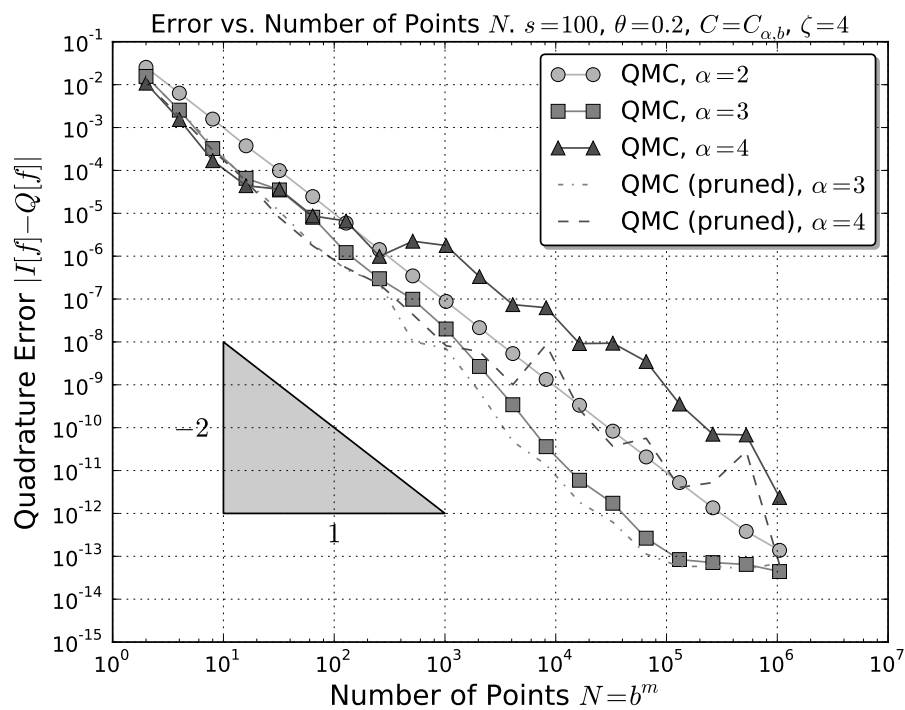


(a) SPOD,  $s = 100$

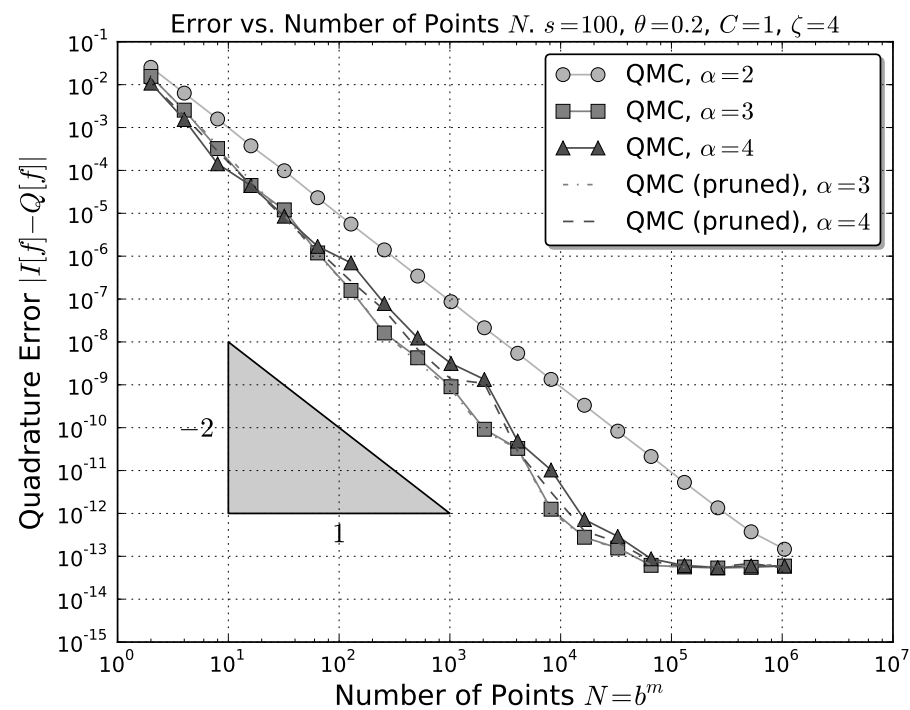


(b) product,  $s = 1000$

Figure 2: CPU time required for the construction of generating vectors of varying order  $\alpha = 2, 3, 4$  for product and SPOD weights vs. the dimension  $s$  in (a) and (b) and vs. the number of points  $N = 2^m$  in (a) and (b).

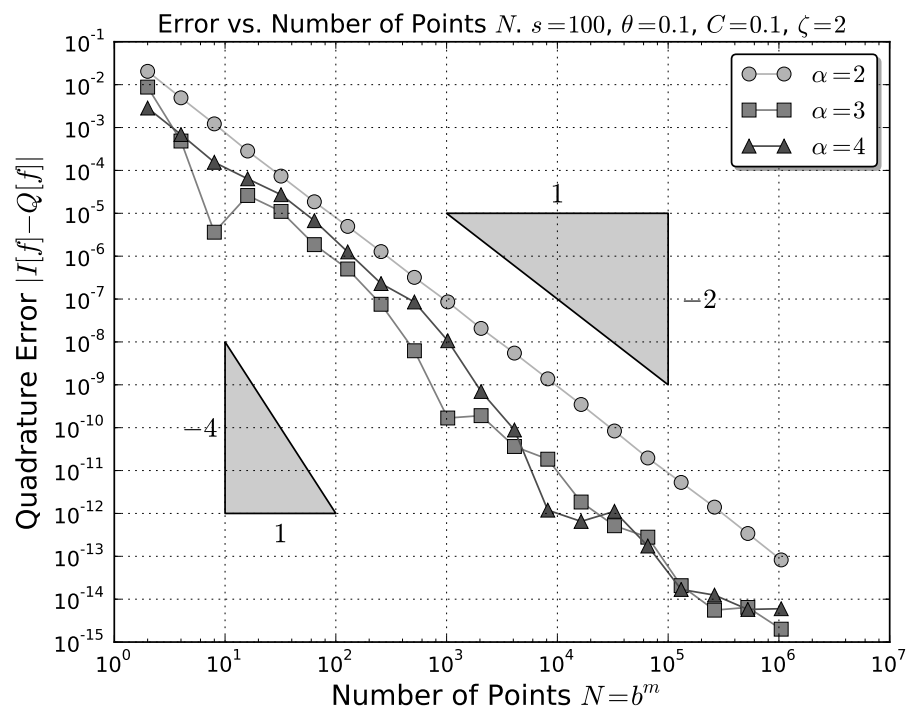


(a)  $C = C_{\alpha,b}$

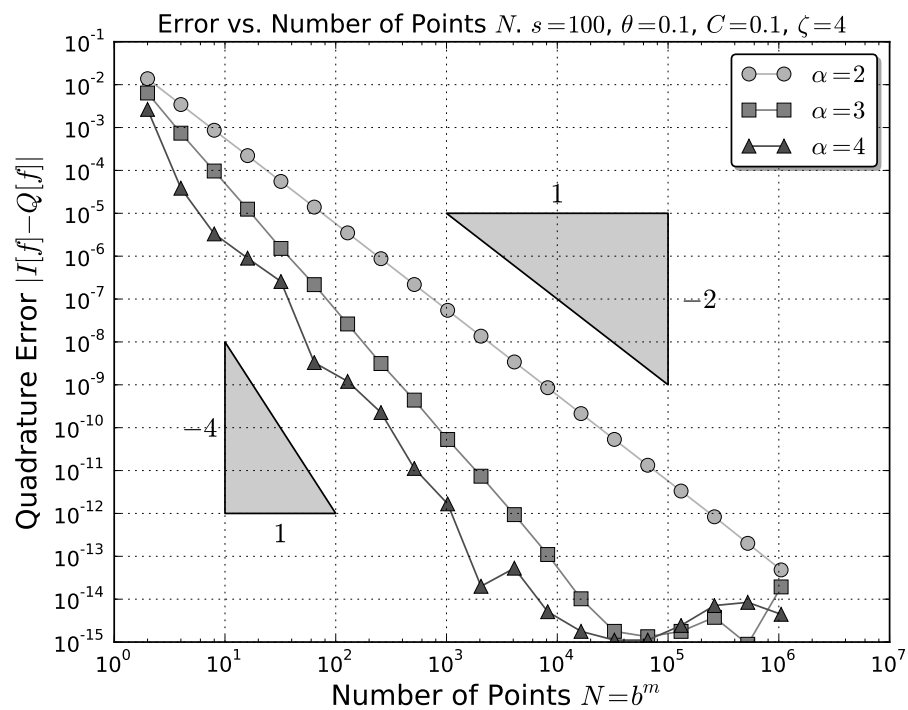


(b)  $C = 1$

Figure 3: Convergence of QMC approximation for SPOD integrand (inv. Karh unen-Loeve) in  $s = 100$  dimensions with interlacing parameter  $\alpha = 2, 3, 4$ , with and without pruning.

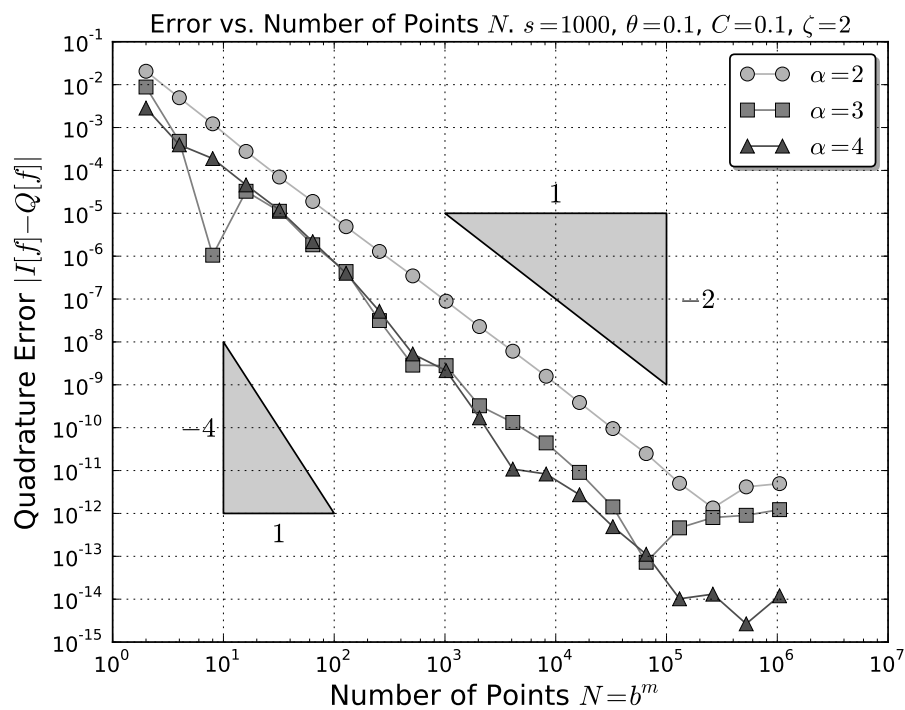


(a)  $s = 100, \zeta = 2$

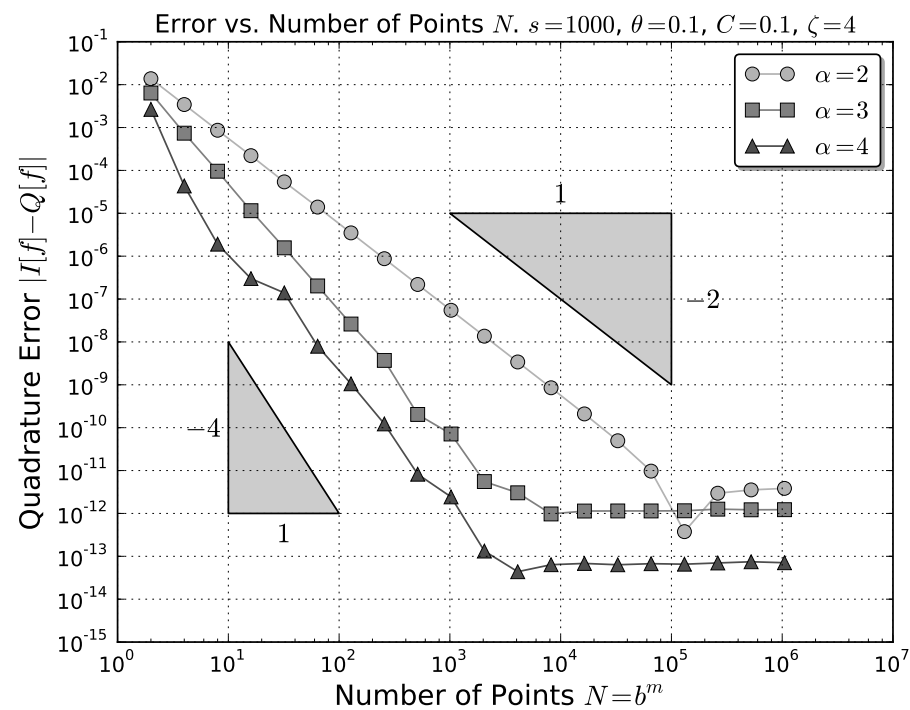


(b)  $s = 100, \zeta = 4$

Figure 4: Convergence of QMC approximation to integral for product weight integrand in  $s = 100, 1000$  dimensions with interlacing parameter  $\alpha = 2, 3, 4$ .

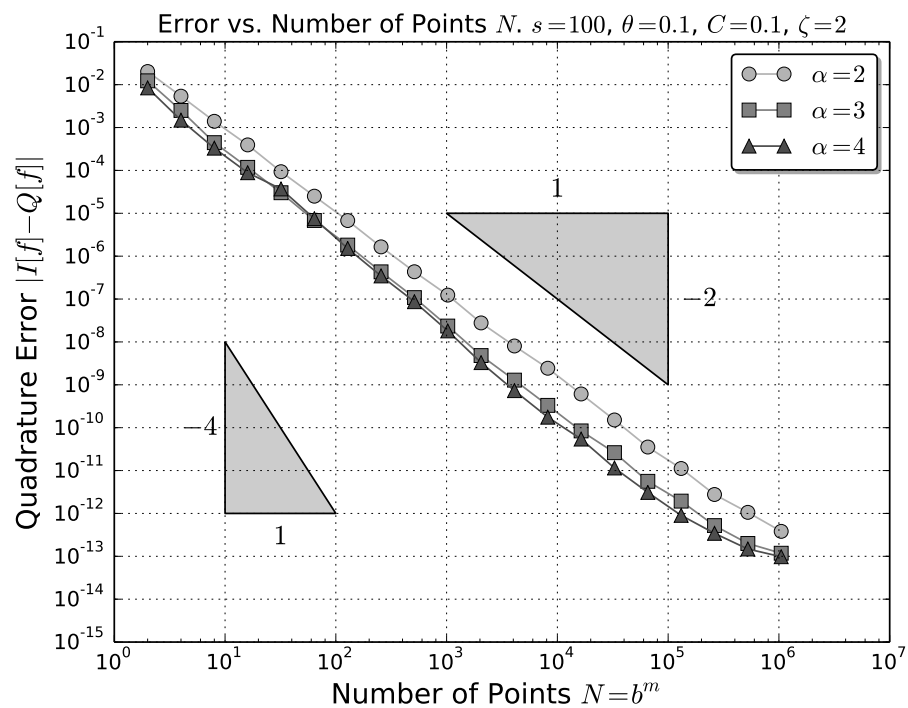


(a)  $s = 1000, \zeta = 2$

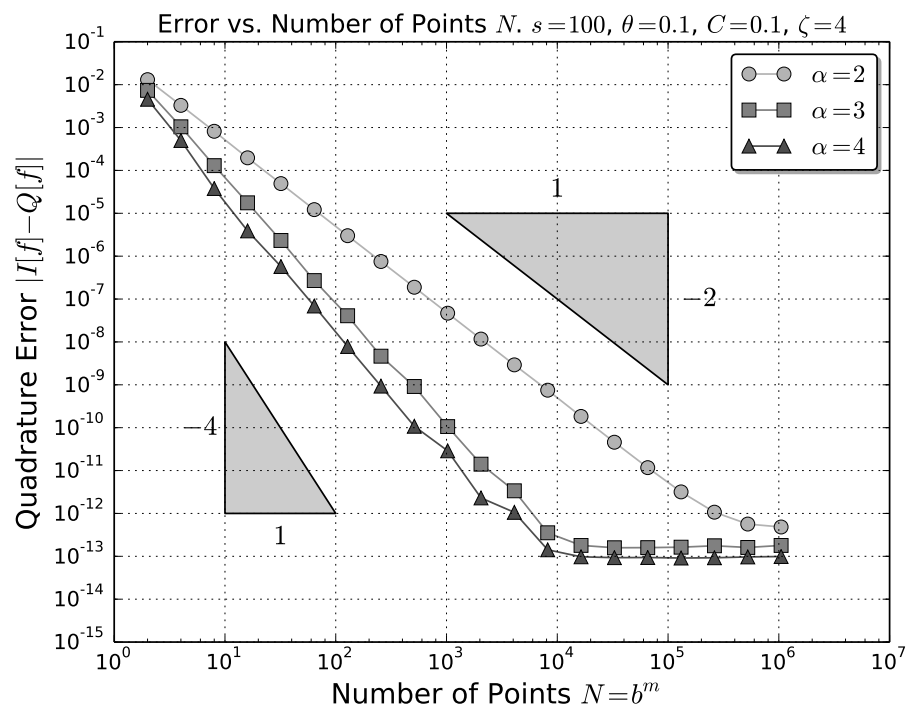


(b)  $s = 1000, \zeta = 4$

Figure 5: Convergence of QMC approximation to integral for product weight integrand in  $s = 100, 1000$  dimensions with interlacing parameter  $\alpha = 2, 3, 4$ .



(a)  $\zeta = 2$



(b)  $\zeta = 4$

Figure 6: Convergence of QMC approximation for SPOD weight integrand in  $s = 100$  dimensions with interlacing parameter  $\alpha = 2, 3, 4$ .

## Conclusions

- Infinite-dimensional, holomorphic-parametric operator equations,
- Advection-Diffusion, Helmholtz in random media, random domains,
- **Sparsity:**
  - affine-parametric equations  $\sum_{j \geq 1} \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X}, \mathcal{X})}^p < \infty$  some  $0 < p < 1$  .
  - holomorphic-parametric equations  $A(\mathbf{y}; \cdot) : \mathcal{X} \mapsto \mathcal{Y}'$  is holomorphic in polydiscs.
- Petrov-Galerkin discretization in  $x \in D$ , Dimension truncation at dimension  $s \in \mathbb{N}$ ,
- $N$  - point QMC quadrature based on digital net of order  $\alpha = 1 + \lfloor 1/p \rfloor$  imply convergence rate

$$|\mathbb{E}[G(u)] - Q_{N,s}[G(u_s^h)]| \leq C(N^{-1/p} + s^{-2(1/p-1)} + h^{t+t'}) .$$

- Work =  $O(N_s M_h)$  analogous to (Single-Level) Monte-Carlo for  $p = 1$ .
- Multi-Level Extension: Report 2014-16, SAM ETH Zürich (Dick, Kuo, LeGia & CS)
- Holomorphic-parametric operator equations: Report 2014-23, SAM ETH Zürich (Dick, Kuo, LeGia & CS).
- Bayesian Inverse Problems:  
for holomorphic-parametric problems, countably-parametric posterior density satisfies derivative bounds  
(Sparsity: Schillings, CS and Stuart 2013, weighted RKHS: Dick, LeGia and CS 2014)  $\implies$   
QMC Rate  $N^{-1/p}$  vs.  $1/2$  for MCMC,  $N^{-(1/p-1)}$  for adaptive Smolyak, CS-based sampling,

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**Thank You.**