High-Order Quasi Monte-Carlo Integration for Bayesian Inversion of Parametric Operator Equations

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Outline

- Infinite-Dimensional Parametric Operator Equations
- Quasi Monte-Carlo Integration and High Order Digital Nets
- Quasi Monte-Carlo Petrov-Galerkin FEM convergence rate
- Bayesian Inverse Problem
- Numerical Results
- Conclusions
- References

Infinite-Dimensional Parametric Operator Equations

- Example: Linear, Affine-Parametric Operator Equation

$$\text{Given} f \in \mathcal{Y}' \;, \quad \text{for every } \boldsymbol{y} \in U \text{ find } u(\boldsymbol{y}) \in \mathcal{X} : \quad A(\boldsymbol{y})\, u(\boldsymbol{y}) = f \;. \tag{1}$$

Here

$$A(\boldsymbol{y}) = A_0 + \sum_{j\geq 1} y_j\, A_j \in \mathcal{L}(\mathcal{X};\mathcal{Y}') \;, \quad \forall \boldsymbol{y} := (y_j)_{j\geq 1} \in U := [-1/2, 1/2]^{\mathbb{N}} \;. \tag{2}$$

- Assumptions:

$$\textbf{Small Fluctuations}: \sum_{j\geq 1} \|A_j\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')} \text{ "small"} \;, \quad \textbf{Sparsity}: \exists 0 < p < 1 : \quad \sum_{j\geq 1} \|A_j\|^p_{\mathcal{L}(\mathcal{X},\mathcal{Y}')} < \infty \;. \tag{3}$$

- Example: Karhúnen-Loeve expansion: $a(x, \boldsymbol{y}) = \bar{a}(x) + \sum_{j\geq 1} y_j \psi_j(x)$

$$A(\boldsymbol{y}) = -\nabla_x \cdot a(x, \boldsymbol{y})\nabla_x = \underbrace{-\nabla_x \cdot \bar{a}(x)\nabla_x}_{A_0} + \sum_{j\geq 1} \underbrace{-\nabla_x \cdot \psi_j(x)\nabla_x}_{A_j} \;, \quad \mathcal{X} = \mathcal{Y} = H^1_0(D) \;.$$

- Goal: given $G \in \mathcal{X}'$, find $\mathbb{E}[G(u(\cdot))]$ with respect to $\boldsymbol{y} \in U$, i.e.,

$$I(G(u)) := \int_U G(u(\boldsymbol{y}))\, \mathrm{d}\boldsymbol{y} \;. \tag{4}$$

Infinite-Dimensional Parametric Operator Equations

- Strategy:

  1. **Dimension Truncation**: truncate (2) to s terms,
  2. solve s-dimensional equation (1) by **Petrov-Galerkin discretization** from $\{\mathcal{X}^h\} \subset \mathcal{X}$,
  3. approximate s-dimensional integral using **QMC integration**,

  $$\frac{1}{N}\sum_{n=0}^{N-1} G\big(u_s^h(\boldsymbol{y}_n - \tfrac{\mathbf{1}}{\mathbf{2}})\big) \ , \qquad (5)$$

     where $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{N-1} \in [0,1]^s$ denote N points from a higher order digital net.
  4. Error bounds explicit w.r.t. to N, h and truncation dimension s.

Infinite-Dimensional Parametric Operator Equations

- Variational Formulation: $\mathfrak{a}_j(\cdot,\cdot) : \mathcal{X} \times \mathcal{Y} \to \mathbb{R}$ via

$$\forall v \in \mathcal{X},\ w \in \mathcal{Y}: \quad \mathfrak{a}_j(v,w) \;=\; {}_{\mathcal{Y}}\langle w, A_j v\rangle_{\mathcal{Y}'}\ , \quad j = 0,1,2,\ldots\ .$$

Assumption:
The sequence $\{A_j\}_{j\geq 0}$ satisfies:

1. the *nominal operator* $A_0 \in \mathcal{L}(\mathcal{X},\mathcal{Y}')$ is boundedly invertible:

$$\inf_{0\neq v\in\mathcal{X}} \sup_{0\neq w\in\mathcal{Y}} \frac{\mathfrak{a}_0(v,w)}{\|v\|_{\mathcal{X}}\|w\|_{\mathcal{Y}}} \geq \mu_0 > 0\ , \quad \inf_{0\neq w\in\mathcal{Y}} \sup_{0\neq v\in\mathcal{X}} \frac{\mathfrak{a}_0(v,w)}{\|v\|_{\mathcal{X}}\|w\|_{\mathcal{Y}}} \geq \mu_0 > 0\ . \tag{6}$$

2. the *fluctuation operators* $\{A_j\}_{j\geq 1}$ are small with respect to A_0: exists $0 < \kappa < 2$ such that

$$\sum_{j\geq 1} \beta_j \leq \kappa < 2\ , \quad \text{where} \quad \beta_j \;:=\; \|A_0^{-1}A_j\|_{\mathcal{L}(\mathcal{X},\mathcal{Y}')}\ , \quad j = 1,2,\ldots\ . \tag{7}$$

Infinite-Dimensional Parametric Operator Equations

Assumption is sufficient for bounded invertibility of $A(\boldsymbol{y})$ **uniformly w.r. to** $\boldsymbol{y} \in U$:
for every realization $\boldsymbol{y} \in U$ of the parameter vector

$$\mathfrak{a}(\boldsymbol{y}; v, w) \;:=\; {}_{\mathcal{Y}}\langle w, A(\boldsymbol{y})v \rangle_{\mathcal{Y}'} \;, \tag{8}$$

satisfies uniform (with respect to $\boldsymbol{y} \in U$) inf-sup conditions: with $\mu = (1 - \kappa/2)\mu_0 > 0$,

$$\forall \boldsymbol{y} \in U : \quad \inf_{0 \neq v \in \mathcal{X}} \sup_{0 \neq w \in \mathcal{Y}} \frac{\mathfrak{a}(\boldsymbol{y}; v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu \;, \quad \inf_{0 \neq w \in \mathcal{Y}} \sup_{0 \neq v \in \mathcal{X}} \frac{\mathfrak{a}(\boldsymbol{y}; v, w)}{\|v\|_{\mathcal{X}} \|w\|_{\mathcal{Y}}} \geq \mu \;. \tag{9}$$

For every $f \in \mathcal{Y}'$ and for every $\boldsymbol{y} \in U$, the parametric problem

$$\text{find} \quad u(\boldsymbol{y}) \in \mathcal{X} : \quad \mathfrak{a}(\boldsymbol{y}; u(\boldsymbol{y}), w) \;=\; {}_{\mathcal{Y}}\langle w, f \rangle_{\mathcal{Y}'} \quad \forall w \in \mathcal{Y} \tag{10}$$

admits unique solution $u(\boldsymbol{y})$ which satisfies the a-priori estimate

$$\|u(\boldsymbol{y})\|_{\mathcal{X}} \;\leq\; \frac{1}{\mu} \|f\|_{\mathcal{Y}'} \;. \tag{11}$$

Infinite-Dimensional Parametric Operator Equations

- Parametric regularity of solutions

$$\forall \boldsymbol{y} \in U : \qquad \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u)(\boldsymbol{y})\|_{\mathcal{X}} \;\leq\; C_0\, |\boldsymbol{\nu}|!\, \boldsymbol{\beta}^{\boldsymbol{\nu}} \|f\|_{\mathcal{Y}'} \quad \text{for all } \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \text{ with } |\boldsymbol{\nu}| < \infty \;, \tag{12}$$

where $0! := 1$, $\boldsymbol{\beta}^{\boldsymbol{\nu}} := \prod_{j\geq 1} \beta_j^{\nu_j}$, with $\beta_j = \|A_0^{-1} A_j\|_{\mathcal{L}(\mathcal{X},\mathcal{X})}$ and $|\boldsymbol{\nu}| = \sum_{j\geq 1} \nu_j$.

- Spatial regularity: *scales of smoothness spaces* $\{\mathcal{X}_t\}_{t\geq 0}$, $\{\mathcal{Y}_t\}_{t\geq 0}$, with

$$\begin{aligned} \mathcal{X} = \mathcal{X}_0 \supset \mathcal{X}_1 \supset \mathcal{X}_2 \supset \cdots \;, \;\; \mathcal{Y} = \mathcal{Y}_0 \supset \mathcal{Y}_1 \supset \mathcal{Y}_2 \supset \cdots \;, \quad \text{and} \\ \mathcal{X}' = \mathcal{X}_0' \supset \mathcal{X}_1' \supset \mathcal{X}_2' \supset \cdots \;, \;\; \mathcal{Y}' = \mathcal{Y}_0' \supset \mathcal{Y}_1' \supset \mathcal{Y}_2' \supset \cdots \;. \end{aligned} \tag{13}$$

- Uniform regularity shift (sufficient for single-level QMC-PG): for $0 < t \leq \bar{t}$,

$$\forall \boldsymbol{y} \in U : \quad f \in \mathcal{Y}_t' \;\Longrightarrow\; \quad u(\boldsymbol{y}) = A(\boldsymbol{y})^{-1} f \in \mathcal{X}_t \;.$$

- Mixed regularity shift (necessary for multi-level QMC-PG):

$$f \in \mathcal{Y}_t' \;\Longrightarrow\; \quad \sup_{\boldsymbol{y}\in U} \|(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} u)(\boldsymbol{y})\|_{\mathcal{X}_t} \;\leq\; C_0\, |\boldsymbol{\nu}|!\, \boldsymbol{\beta}_t^{\boldsymbol{\nu}} \|f\|_{\mathcal{Y}_t'} \quad \text{for all } \boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} \text{ with } |\boldsymbol{\nu}| < \infty \;.$$

Infinite-Dimensional Parametric Operator Equations

- Best N-term polynomial chaos approximation
  Under Assumption, $u(\boldsymbol{y}) : U \to \mathcal{X}$ can be expanded in (unconditionally convergent in $L^2(U; d\boldsymbol{y})$) gpc series

$$\forall \boldsymbol{y} \in U : \qquad u(\boldsymbol{y}) = \sum_{\mathcal{F}} u_{\boldsymbol{\nu}} L_{\boldsymbol{\nu}}(\boldsymbol{y}) \; , \qquad \text{where} \quad u_{\boldsymbol{\nu}} = (u, L_{\boldsymbol{\nu}})_{L^2(U;d\boldsymbol{y})} \in \mathcal{X} \; .$$

  Here $\mathcal{F} = \{\boldsymbol{\nu} \in \mathbb{N}_0^{\mathbb{N}} : |\boldsymbol{\nu}| < \infty\}$, and $L_{\boldsymbol{\nu}}$ is the ($L^2(U; d\boldsymbol{y})$-normalized) tensorized Legendre polynomial

$$\forall \boldsymbol{\nu} \in \mathcal{F}, \forall \boldsymbol{y} \in U : \quad L_{\boldsymbol{\nu}}(\boldsymbol{y}) := \prod_{j \geq 1} L_{\nu_j}(y_j) \qquad (\text{note } \; L_0 \equiv 1) \; .$$

- [Cohen & DeVore & CS (2011), (Chkifa & Cohen & CS 2014)]
  Assume that $\boldsymbol{\beta} \in \ell^p(\mathbb{N})$ for some $0 < p < 1$. Then for every $N \in \mathbb{N}$ exists $\Lambda \subset \mathcal{F}$ such that, for $q = 2, \infty$

$$\#(\Lambda) = N \qquad \text{and} \qquad \|u - u_\Lambda\|_{L^q(U,d\boldsymbol{y};\mathcal{X})} \leq C(q) N^{-(1/p - 1/q')} \; .$$

  q' conjugate of $q = 2, \infty$, constant $C > 0$ independent of N and of dimension.

- Proof nonconstructive. "Constructive versions" (Chkifa & Cohen & DeVore & CS 2012-2014).

Infinite-Dimensional Parametric Operator Equations

- *Petrov-Galerkin discretization*:
  Let $\{\mathcal{X}^h\}_{h>0} \subset \mathcal{X}$ and $\{\mathcal{Y}^h\}_{h>0} \subset \mathcal{Y}$ dense families of subspaces in $\mathcal{X}$ and $\mathcal{Y}$.

- *Approximation Properties*: for $0 < t \leq \bar{t}$ and $0 < t' \leq \bar{t}'$, and for $0 < h \leq h_0$, there hold

$$\begin{aligned} \forall v \in \mathcal{X}_t \ &: \quad \inf_{v^h \in \mathcal{X}^h} \|v - v^h\|_{\mathcal{X}} \ \leq \ C_t \, h^t \, \|v\|_{\mathcal{X}_t} \ , \\ \forall w \in \mathcal{Y}_{t'} \ &: \quad \inf_{w^h \in \mathcal{Y}^h} \|w - w^h\|_{\mathcal{Y}} \ \leq \ C_{t'} \, h^{t'} \, \|w\|_{\mathcal{Y}_{t'}} \ . \end{aligned} \tag{14}$$

$$\forall \, 0 \leq t \leq \bar{t} : \quad \sup_{\boldsymbol{y} \in U} \|A(\boldsymbol{y})^{-1}\|_{\mathcal{L}(\mathcal{Y}'_t, \mathcal{X}_t)} < \infty \ . \tag{15}$$

- *Stability*: assume that $(\mathcal{X}^h, \mathcal{Y}^h)$ satisfy discr. inf-sup condition for A_0.
  Then there hold uniform (with respect to $\boldsymbol{y} \in U$) discrete inf-sup conditions

$$\forall \boldsymbol{y} \in U : \quad \inf_{0 \neq v^h \in \mathcal{X}^h} \sup_{0 \neq w^h \in \mathcal{Y}^h} \frac{\mathfrak{a}(\boldsymbol{y}; v^h, w^h)}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 \ , \tag{16}$$

$$\forall \boldsymbol{y} \in U : \quad \inf_{0 \neq w^h \in \mathcal{Y}^h} \sup_{0 \neq v^h \in \mathcal{X}^h} \frac{\mathfrak{a}(\boldsymbol{y}; v^h, w^h)}{\|v^h\|_{\mathcal{X}} \|w^h\|_{\mathcal{Y}}} \geq \bar{\mu} > 0 \ . \tag{17}$$

Infinite-Dimensional Parametric Operator Equations

- For every $0 < h \leq h_0$ and for every $\boldsymbol{y} \in U$, Petrov-Galerkin approximation

$$\text{find } u^h(\boldsymbol{y}) \in \mathcal{X}^h : \quad \mathfrak{a}(\boldsymbol{y}; u^h(\boldsymbol{y}), w^h) = {}_{\mathcal{Y}}\langle w^h, f \rangle_{\mathcal{Y}'} \quad \forall w^h \in \mathcal{Y}^h \,, \tag{18}$$

admits a unique solution $u^h(\boldsymbol{y})$ which satisfies the a-priori estimate

$$\|u^h(\boldsymbol{y})\|_{\mathcal{X}} \leq \frac{1}{\bar{\mu}} \|f\|_{\mathcal{Y}'} \,. \tag{19}$$

- Quasioptimality: exists a constant $C > 0$ such that for all $\boldsymbol{y} \in U$

$$\|u(\boldsymbol{y}) - u^h(\boldsymbol{y})\|_{\mathcal{X}} \leq \frac{C}{\bar{\mu}} \inf_{0 \neq v^h \in \mathcal{X}^h} \|u(\boldsymbol{y}) - v^h\|_{\mathcal{X}} \,. \tag{20}$$

- *Convergence Rate*: Ex. $C > 0$ such that for every $f \in \mathcal{Y}'_t$ with $0 < t \leq \bar{t}$ as $h \to 0$

$$\|u(\boldsymbol{y}) - u^h(\boldsymbol{y})\|_{\mathcal{X}} \leq C\, h^t \, \|f\|_{\mathcal{Y}'_t} \,. \tag{21}$$

- Dimension-Truncation: For $\boldsymbol{y} \in U$ and $s \in \mathbb{N}$, $(y_1, y_2, ..., y_s, 0, 0, ...) \in U$, so

all bounds valid for $[-1/2, 1/2]^s$ uniformly w.r. to s.

Infinite-Dimensional Parametric Operator Equations

- *Regularity*:

$$\exists\, 0 < t' \leq \bar{t} : \quad G(\cdot) \in \mathcal{X}'_{t'} \,, \tag{22}$$

- *Adjoint Regularity*: exists $C_{t'} > 0$ such that for every $\boldsymbol{y} \in U$,

$$\forall \boldsymbol{y} \in U : \quad w(\boldsymbol{y}) = (A^*(\boldsymbol{y}))^{-1} G \in \mathcal{Y}_{t'} \,, \quad \|w(\boldsymbol{y})\|_{\mathcal{Y}_{t'}} \,\leq\, C_{t'} \, \|G\|_{\mathcal{X}'_{t'}} \,. \tag{23}$$

- *Superconvergence* (Aubin-Nitsche):
for every $f \in \mathcal{Y}'_t$ with $0 < t \leq \bar{t}$, for every $G(\cdot) \in \mathcal{X}'_{t'}$ with $0 < t' \leq \bar{t}$

$$\sup_{\boldsymbol{y} \in U} \left| G(u(\boldsymbol{y})) - G(u^h(\boldsymbol{y})) \right| \,\leq\, C \, h^{\tau} \, \|f\|_{\mathcal{Y}'_t} \, \|G\|_{\mathcal{X}'_{t'}} \,. \tag{24}$$

where $0 < \tau := t + t' \leq 2\bar{t}$.

Dimension Truncation

Assume the A_j are enumerated s.t. $\beta_j := \|A_0^{-1}A_j\|_{\mathcal{X}}$ satisfy

$$\beta_1 \geq \beta_2 \geq \cdots \geq \beta_j \geq \cdots \ . \tag{25}$$

Then, for every $f \in \mathcal{Y}'$, every $\boldsymbol{y} \in U$ and for every $s \in \mathbb{N}$,

$$\sup_{\boldsymbol{y}\in U} \|u(\boldsymbol{y}) - u_s(\boldsymbol{y})\|_{\mathcal{X}} \ \leq \ \frac{C}{\mu}\,\|f\|_{\mathcal{Y}'} \sum_{j\geq s+1} \beta_j \ . \tag{26}$$

For every $G(\cdot) \in \mathcal{X}'$,

$$|I(G(u)) - I(G(u_s))| \ \leq \ \frac{\tilde{C}}{\mu}\,\|f\|_{\mathcal{Y}'}\,\|G\|_{\mathcal{X}'} \left(\sum_{j\geq s+1} \beta_j\right)^2 \tag{27}$$

for $\tilde{C} > 0$ independent of s, f and G.

If conditions (25) hold, for any $0 < p < 1$ and for any $s \in \mathbb{N}$ holds the dimension-truncation error bound

$$\sum_{j\geq s+1} \beta_j \ \leq \ \min\left(\frac{1}{1/p - 1}, 1\right) \left(\sum_{j\geq 1} \beta_j^p\right)^{1/p} s^{-(1/p-1)} \ .$$

Quasi Monte-Carlo Integration and High Order Digital Nets

- Consider *general* s-variate integrand $F \in C^0([0,1]^s)$. Approximate s-dimensional integral

$$I_s(F) := \int_{[0,1]^s} F(\boldsymbol{y}) \, \mathrm{d}\boldsymbol{y} \tag{28}$$

  where $F(\boldsymbol{y}) = G(u_s^h(\boldsymbol{y} - \frac{\mathbf{1}}{\mathbf{2}}))$ by

- N-point QMC method: an equal-weight quadrature rule

$$Q_{N,s}(F) := \frac{1}{N} \sum_{n=0}^{N-1} F(\boldsymbol{y}_n) \, , \tag{29}$$

  with judiciously chosen points $\boldsymbol{y}_0, \ldots, \boldsymbol{y}_{N-1} \in [0,1]^s$.

Quasi Monte-Carlo Integration and High Order Digital Nets

Theorem 1 [Dick, Gia, Kuo, Nuyens, CS 2013]
Let $s \geq 1$ and $N = b^m$ for $m \geq 1$ and prime b. Let $\boldsymbol{\beta} = (\beta_j)_{j\geq 1}$ be a sequence of positive numbers s.t.

$$\exists\, 0 < p < 1 : \quad \sum_{j=1}^{\infty} \beta_j^p < \infty \; . \tag{30}$$

Define $\boldsymbol{\beta}_{\{1:s\}} = (\beta_j)_{1\leq j\leq s}$ and

$$\alpha \;:=\; \lfloor 1/p \rfloor + 1 \; . \tag{31}$$

Assume for every $s \in \mathbb{N}$, for every $\boldsymbol{y} \in [-1/2, 1/2]^{\mathbb{N}}$

$$\forall\, \boldsymbol{\nu} \in \{0, 1, \ldots, \alpha\}^s : \quad |(\partial_{\boldsymbol{y}}^{\boldsymbol{\nu}} F)(\boldsymbol{y}_{\{1:s\}})| \;\leq\; c\, |\boldsymbol{\nu}|!\, \boldsymbol{\beta}_{\{1:s\}}^{\boldsymbol{\nu}} \tag{32}$$

Then, an interlaced polynomial lattice rule of order $\alpha \geq 1$ with N points can be constructed using a fast component-by-component algorithm, with $\mathbf{cost}$ $\mathcal{O}(\alpha\, s\, N \log N + \alpha^2\, s^2 N)$ $\mathbf{operations}$, such that

$$|I_s(F) - Q_{N,s}(F)| \;\leq\; C_{\alpha,\boldsymbol{\beta},b,p}\, N^{-1/p} \; ,$$

where $C_{\alpha,\boldsymbol{\beta},b,p} < \infty$ is independent of s and N.

Combined Error Bound

Theorem 2 [Single-Level High-Order Quasi Monte-Carlo Galerkin Error bound]

1. Approximate $I(G(u(\cdot)))$ by dimension-truncation and interlaced polynomial lattice rule (5)

$$\text{order } \alpha = \lfloor 1/p \rfloor + 1 \;, \;\; \text{with} N = b^m \;\text{ points in } s \text{ dimensions,}$$

   with Petrov-Galerkin discretization in D with subspace $\mathcal{X}^h$ with $M_h = \dim(\mathcal{X}^h)$ DoF, cost $\mathcal{O}(M_h)$.
   Then ex. $C > 0$ independent of s, h and N such that with $\tau = t + t'$

$$|I(G(u)) - Q_{N,s}(G(u_s^h))| \;\leq\; C \left((s^{-2(1/p-1)} + N^{-1/p}) \|f\|_{\mathcal{Y}'} \, \|G(\cdot)\|_{\mathcal{X}'} + h^{\tau} \, \|f\|_{\mathcal{Y}'_t} \|G(\cdot)\|_{\mathcal{X}'_{t'}} \right) \;.$$

2. Cost for evaluation of $Q_{N,s}(G(u_s^h))$ is $\mathcal{O}(sNM_h)$ operations.
3. Cost for CBC construction of the interlaced polynomial lattice rule

$$\mathcal{O}(\alpha\, s\, N \log N + \alpha^2 s^2 N) \text{ operations,} \qquad \mathcal{O}(\alpha\, s\, N) \;\text{ memory .}$$

Proof

$$I(G(u)) - Q_{N,s}(G(u_s^h)) = [I(G(u)) - I(G(u_s))] + [I(G(u_s)) - I(G(u_s^h))] + [I(G(u_s^h)) - Q_{N,s}(G(u_s^h))] \;.$$

$\square$

Bayesian Inverse Problems

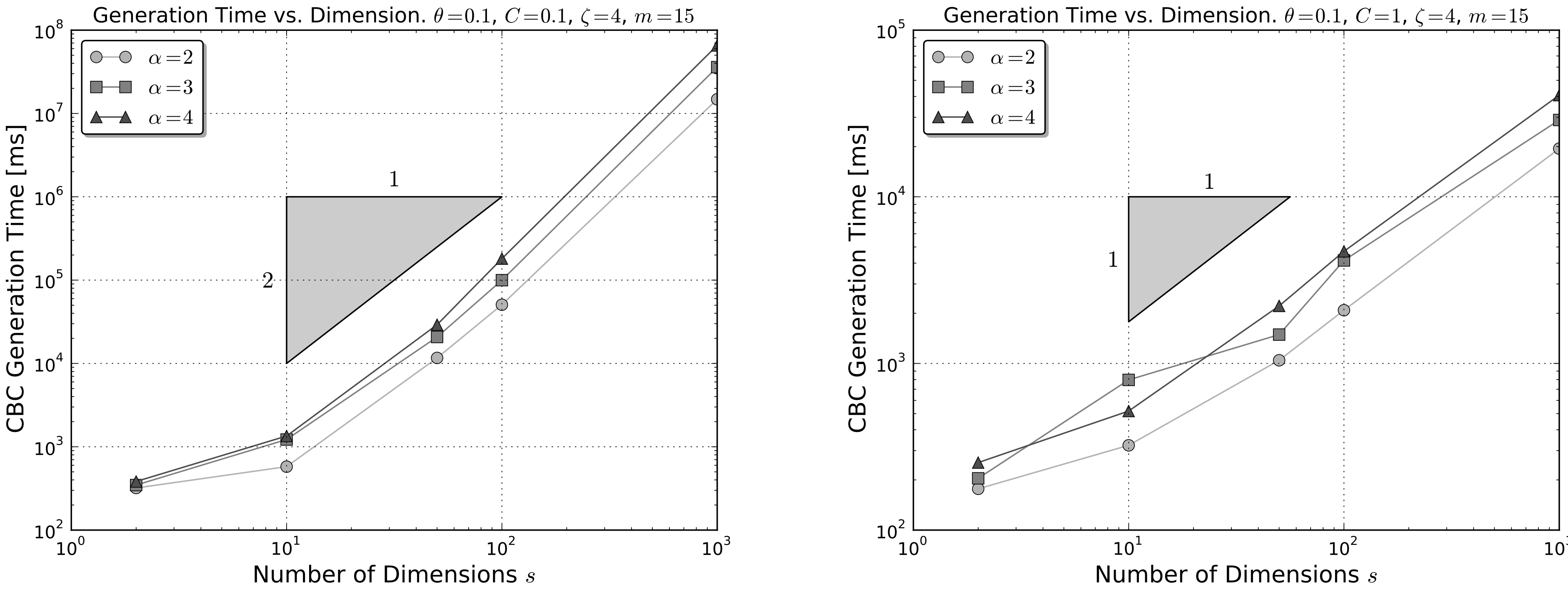
- uncertain input data $u(\boldsymbol{y}) = \langle u \rangle + \sum_{j\geq 1} y_j \psi_j \in X$, uniform prior π_0 on u,
- forward equation: $A(u;q) = f$, $f \in \mathcal{Y}'$ known, forward solution map $G(u(\boldsymbol{y}))$ holomorphic w.r. to $\boldsymbol{y}$.
- noisy observation data: $\delta = \mathcal{G}(u) + \eta \in Y$, $\mathcal{G} = \mathcal{O} \circ G : X \to Y$, $\eta \sim N(0,\Gamma) \in Y = \mathbb{R}^K$,
- QoI: $\phi(u)$.

Bayes' Estimate:

$$I_\Gamma(\phi(u)) := \frac{1}{Z} \int_{\boldsymbol{y}\in[-1/2,1/2]^{\mathbb{N}}} \underbrace{\exp\left(-\frac{1}{\Gamma}\|\delta - \mathcal{G}(u(\boldsymbol{y}))\|_Y^2\right) \phi(u(\boldsymbol{y}))}_{=:F(\boldsymbol{y})} \pi_0(d\boldsymbol{y})$$

Theorem 3 [High-Order Quasi Monte-Carlo for Bayesian Inverse Problems]
For every $\Gamma > 0$, the parametric, deterministic integrand function $F(\boldsymbol{y})$ satisfies the derivative bounds (32).
The preceding error bounds apply to Bayesian Estimation.

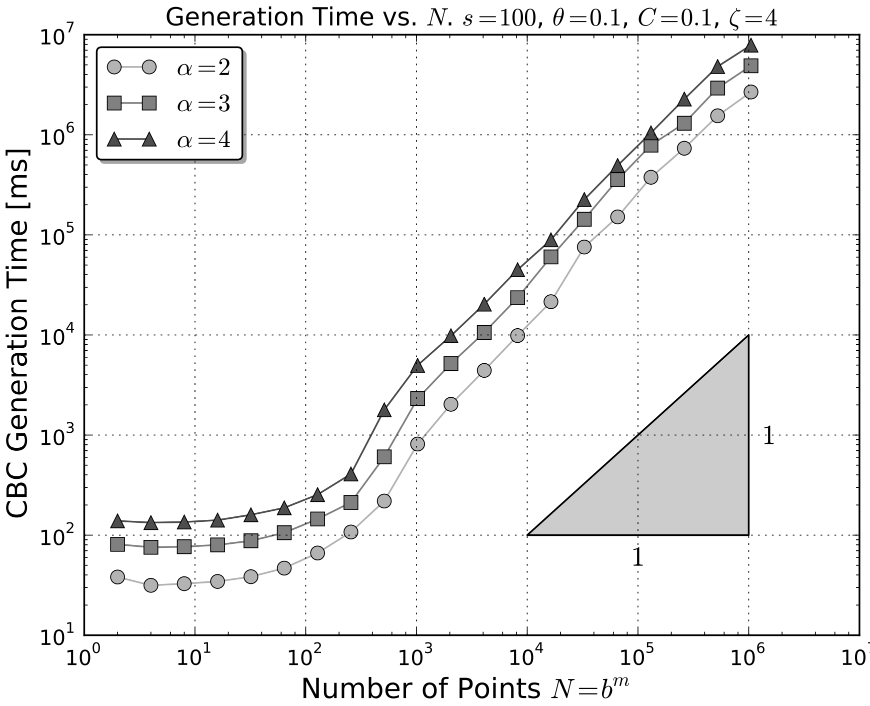
Proof: Analytic continuation and Cauchy's Theorem. $\square$



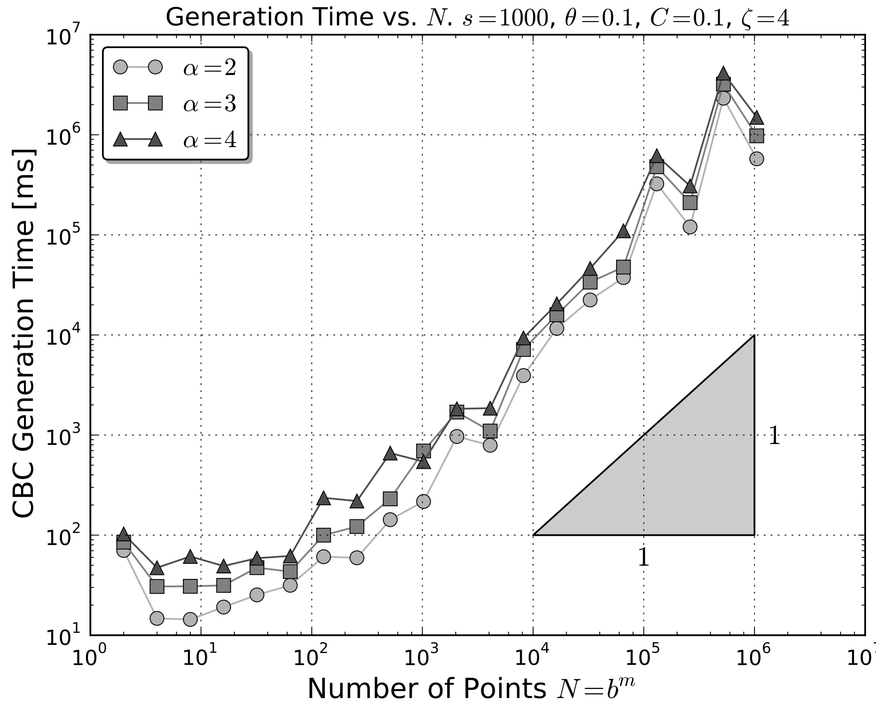
(a) SPOD, $C = 0.1$

(b) product, $C = 1$

Figure 1: CPU time required for the construction of generating vectors of varying order $\alpha = 2, 3, 4$ for product and SPOD weights vs. the dimension s in (a) and (b) and vs. the number of points $N = 2^m$ in (a) and (b).

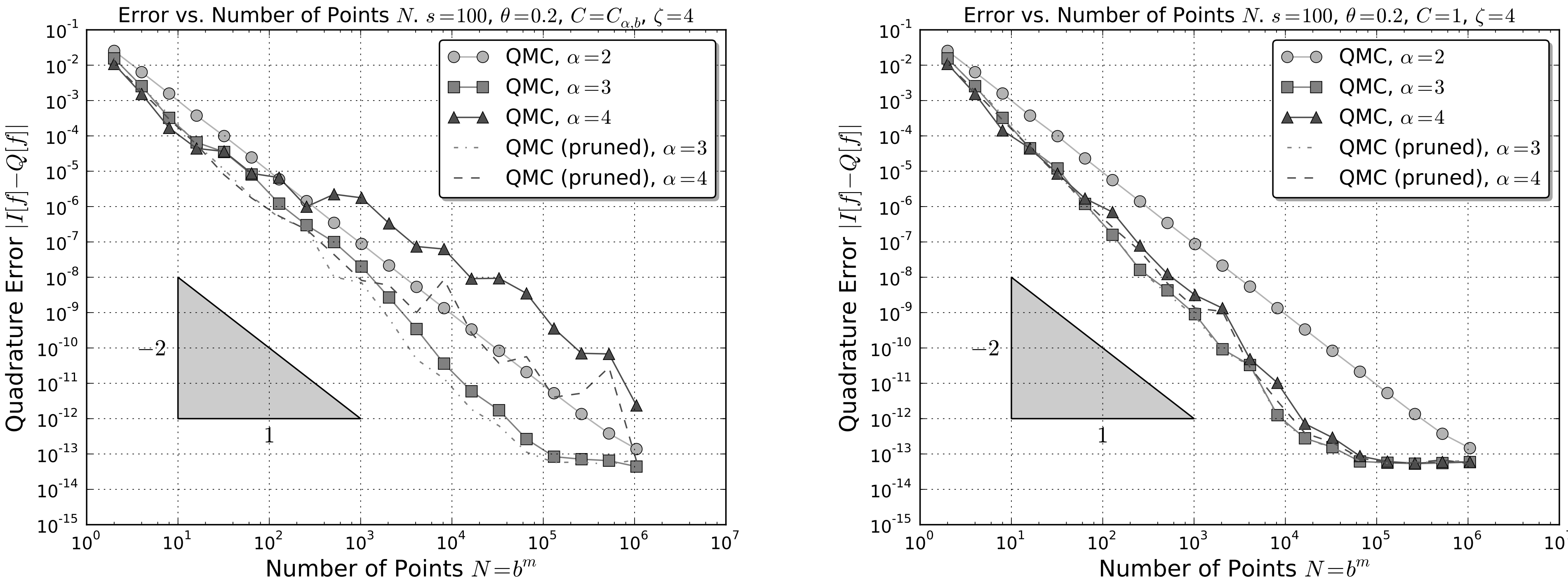


(a) SPOD, $s = 100$



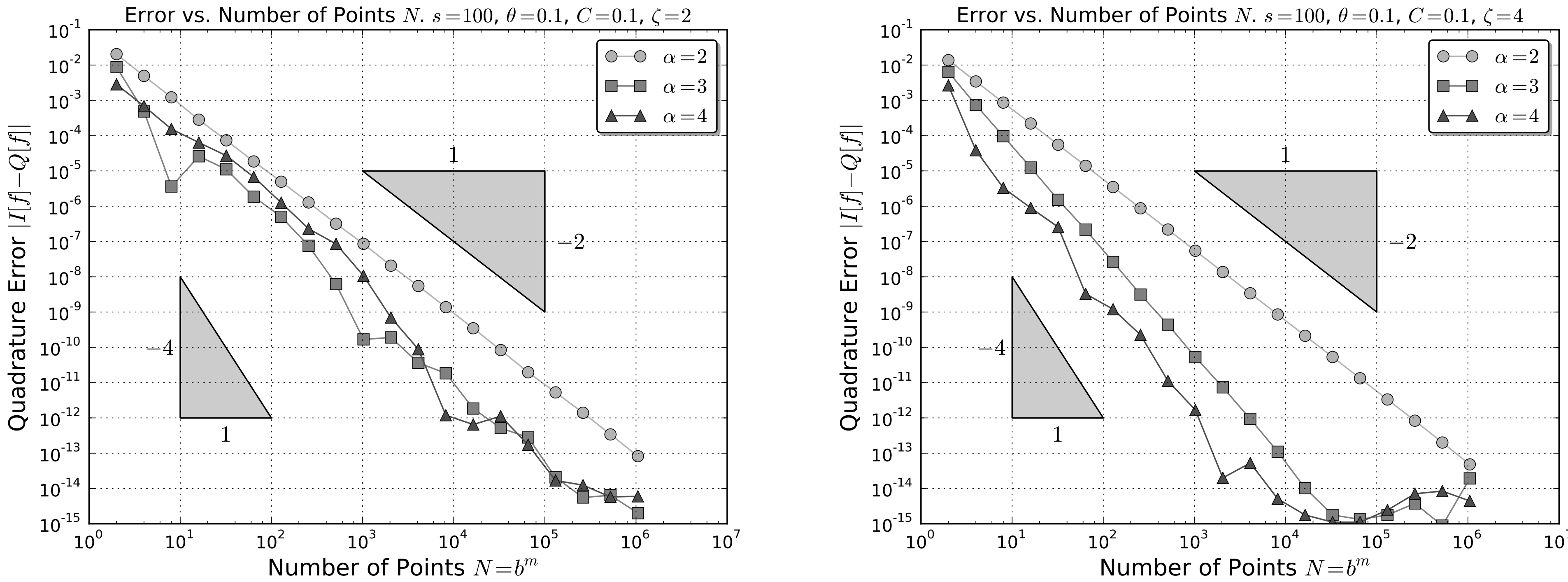
(b) product, $s = 1000$

Figure 2: CPU time required for the construction of generating vectors of varying order $\alpha = 2, 3, 4$ for product and SPOD weights vs. the dimension s in (a) and (b) and vs. the number of points $N = 2^m$ in (a) and (b).



(a) $C = C_{\alpha,b}$ (b) $C = 1$

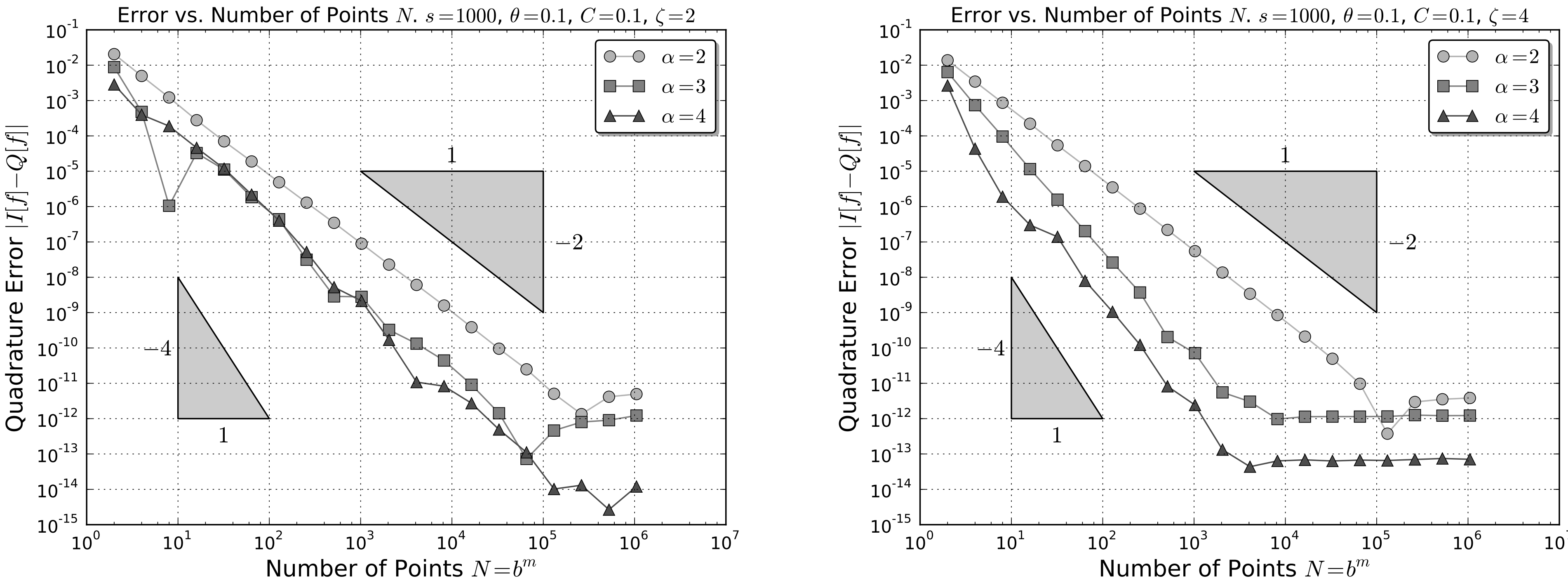
Figure 3: Convergence of QMC approximation for SPOD integrand (inv. Karhúnen-Loeve) in $s = 100$ dimensions with interlacing parameter $\alpha = 2, 3, 4$, with and without pruning.



(a) $s = 100$, $\zeta = 2$

(b) $s = 100$, $\zeta = 4$

Figure 4: Convergence of QMC approximation to integral for product weight integrand in $s = 100, 1000$ dimensions with interlacing parameter $\alpha = 2, 3, 4$.



(a) $s = 1000$, $\zeta = 2$

(b) $s = 1000$, $\zeta = 4$

Figure 5: Convergence of QMC approximation to integral for product weight integrand in $s = 100, 1000$ dimensions with interlacing parameter $\alpha = 2, 3, 4$.

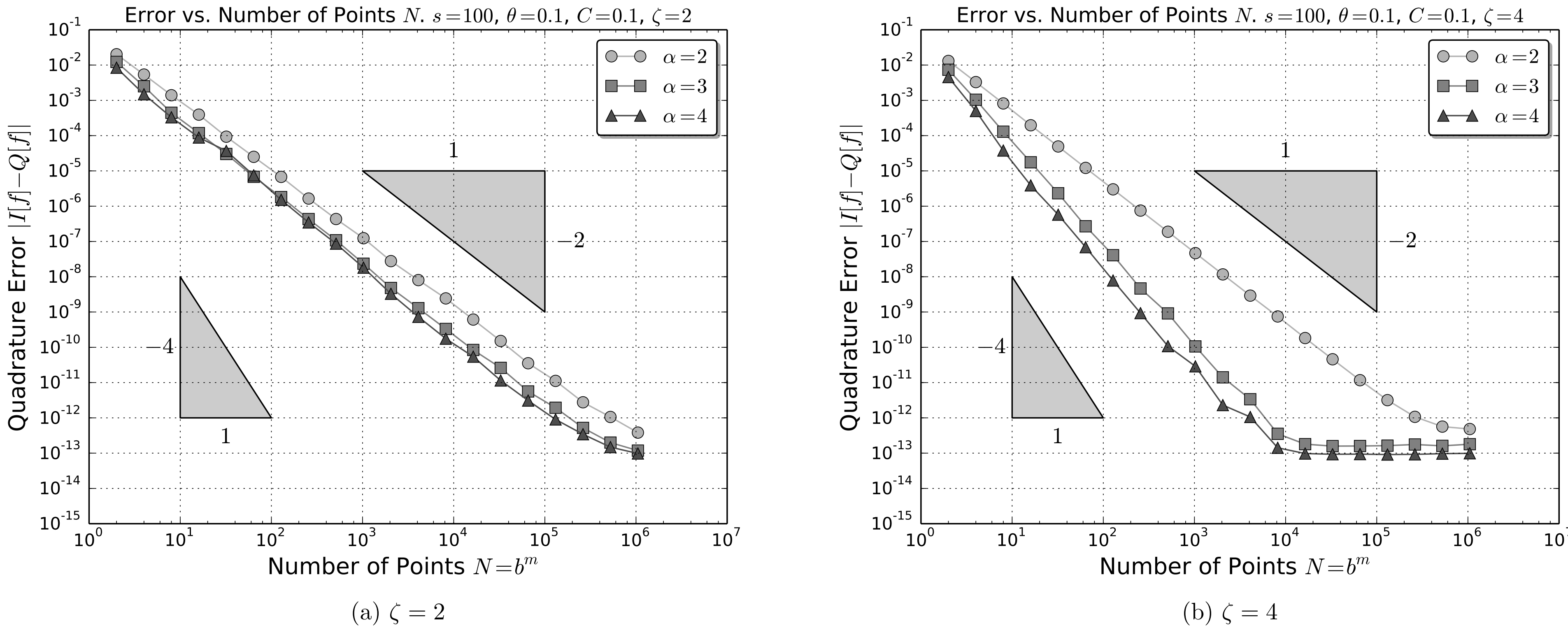


Figure 6: Convergence of QMC approximation for SPOD weight integrand in $s = 100$ dimensions with interlacing parameter $\alpha = 2, 3, 4$.

Conclusions

- Infinite-dimensional, holomorphic-parametric operator equations,
- Advection-Diffusion, Helmholtz in random media, random domains,
- **Sparsity**:
  - affine-parametric equations $\sum_{j\geq 1} \|A_0^{-1}A_j\|^p_{\mathcal{L}(\mathcal{X},\mathcal{X})} < \infty$ some $0 < p < 1$.
  - holomorphic-parametric equations $A(\boldsymbol{y};\cdot) : \mathcal{X} \mapsto \mathcal{Y}'$ is holomorphic in polydiscs.
- Petrov-Galerkin discretization in $x \in D$, Dimension truncation at dimension $s \in \mathbb{N}$,
- N - point QMC quadrature based on digital net of order $\alpha = 1 + \lfloor 1/p \rfloor$ imply convergence rate

$$\left|\mathbb{E}[G(u)] - Q_{N,s}[G(u_s^h)]\right| \leq C(N^{-1/p} + s^{-2(1/p-1)} + h^{t+t'}) \ .$$

- Work $= O(NsM_h)$ analogous to (Single-Level) Monte-Carlo for $p = 1$.
- Multi-Level Extension: Report 2014-16, SAM ETH Zürich (Dick, Kuo, LeGia & CS)
- Holomorphic-parametric operator equations: Report 2014-23, SAM ETH Zürich (Dick, Kuo, LeGia & CS).
- Bayesian Inverse Problems:
  for holomorphic-parametric problems, countably-parametric posterior density satisfies derivative bounds (Sparsity: Schillings, CS and Stuart 2013, weighted RKHS: Dick, LeGia and CS 2014) $\Longrightarrow$
  QMC Rate $N^{-1/p}$ vs. $1/2$ for MCMC, $N^{-(1/p-1)}$ for adaptive Smolyak, CS-based sampling,

References

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Thank You.