Experiments on Siegel modular forms of genus 2 (Not only on the Paramodular Conjecture)

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Modular Forms and Curves of Low Genus: Computational Aspects
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THIS IS
YOUR
LIFE

Experiments with L-functions of Siegel modular forms

1. Compute a basis for the space of Siegel modular forms of genus 2 and identify the Hecke eigenforms.
2. Compute (a lot of) coefficients of the Hecke eigenforms.
3. Compute the Hecke eigenvalues of the Hecke eigenforms.
4. Compute the Euler factors of the L-function and therefore the Dirichlet series.
5. Evaluate the L-function at a point s.

Why compute Siegel modular forms and their L-functions?

- Verify conjectures...
- Formulate conjectures...
- Discovering unexpected phenomena...
- To understand abstract things concretely...
- Because we can...

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Harder's Conjecture

- Generalizes Ramanujan's congruence

$$\tau(p) \equiv p^{11} + 1 \pmod{691}.$$

- Let $f \in S_r^{(1)}$ be a Hecke eigenform with coefficient field $\mathbb{Q}_f$ and let ℓ be an ordinary prime in $\mathbb{Q}_f$ (i.e. such that the ℓ-th Hecke eigenvalue of f is not divisible by ℓ). Suppose $s \in \mathbb{N}$ is such that ℓ^s divides the algebraic critical value $\tilde{\Lambda}(f, t)$. Then there exists a Hecke eigenform $F \in S_{k,j}^{(2)}$, where $k = r - t + 2$, $j = 2t - r - 2$, such that

$$\mu_{p^\delta}(F) \equiv \mu_{p^\delta}(f) + p^{\delta(k+j-1)} + p^{\delta(k-2)} \pmod{\ell^s}$$

for all prime powers p^δ.

Harder's Conjecture

- In joint work with Ghitza and Sulon we verified the conjecture computationally for $r \leq 60$ and for

$$p^\delta \in \{2, 3, 4, 5, 7, 8, 9, 11, 13, 17, 19, 23, 25, 27, 29, 31, 125\}.$$

- A variant of Harder's conjecture due to Bergström, Faber, van der Geer, and Harder involves critical values of the symmetric square L-function. We verified this conjecture for $r \leq 32$ and roughly the same list of prime powers.

Our computations were in weight $(2, j)$.

Maeda's Conjecture

- Let $[T_p]$ be the matrix of the Hecke operator on the space of modular forms of weight k and level 1. It has been conjectured that the characteristic polynomial of this matrix is irreducible.
- For Siegel modular forms, the first weight at which the space becomes two-dimensional, the characteristic polynomial factors into linear factors. In weights 24 and 26 we have these "terrifying example[s] due to Skoruppa".
- As we verified Harder's conjecture, we found terrifying examples of vector valued Siegel modular forms in weights $(k, 2)$.

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Rankin convolution

- A Siegel modular form F has a Fourier expansion indexed by positive semidefinite binary quadratic forms. If we gather the coefficients in a certain way, we can write

$$F(z,\tau,z') = \sum_{n\geq 0} \phi_{F,n}(z,\tau)q'^n$$

  where each $\phi_{F,n}$ is a Jacobi form of the same weight and of index n.
- For two modular forms F and G define the convolution Dirichlet series:

$$D_{F,G}(s) = \zeta(2s-2k+4)\sum_{n\geq 1}\langle \phi_{G,n}, \phi_{F,n}\rangle n^{-s},$$

  where $\langle \cdot, \cdot \rangle$ is the Petersson inner product of two Jacobi forms.

Rankin convolution

- A Theorem due to Skoruppa and Zagier: if F is a Siegel modular form and G is a Saito-Kurokawa lift, then

$$D_{F,G}(s) = \langle \phi_{F,1}, \phi_{G,1} \rangle L(F, s)$$

  where $L(F, s)$ is the spin L-function of F.
- In joint work with Skoruppa and Strömberg, we asked what if G is not a lift?
- We identified all the eigenforms in weights between 20 and 30 and used those to compute the Jacobi forms used in the computations of $D_{F,G}(s)$.

Rankin convolution

- We implemented a method to compute the Petersson inner product.
- We computed $D_{F,G}(s)$ for all Hecke eigenforms F, G of the same weight k for $20 \leq k \leq 30$.
- We showed that the Dirichlet series $D_{F,G}(s)$ was not an L-function: its coefficients weren't even multiplicative!

Formulating Böcherer's Conjecture in the paramodular setting

For a fundamental discriminant $D < 0$ coprime to the level, Böcherer's Conjecture states:

$$L(F, 1/2, \chi_D) = C_F |D|^{1-k} A(D)^2$$

where F is a Siegel modular form of weight k, $C_F > 0$ is a constant that only depends on F, and $A(D)$ is an average of the coefficients of F of discriminant D.

Putting the Conjecture in context:

- It's a generalization of Waldspurger's formula relating central values of elliptic curve L-functions to sums of coefficients of half-integer weight modular forms.
- In general, computing coefficients of Siegel modular forms is much easier than computing their Hecke eigenvalues (and therefore their L-functions). So this formula would provide a computationally feasible way to compute lots of central values.
- A theorem of Saha states that a weak version of the conjecture implies multiplicity one for Siegel modular forms of level 1.

The state of the art:

- Böcherer originally proved it for Siegel modular forms that are Saito-Kurokawa lifts.
- Kohnen and Kuss verified the conjecture numerically for the first few rational Siegel modular eigenforms that are not lifts (these are in weight 20-26) for only a few fundamental discriminants.
- Raum (very) recently verified the conjecture numerically for nonrational Siegel modular eigenforms that are not lifts for a few more fundamental discriminants.
- Böcherer and Schulze-Pillot formulated a conjecture for Siegel modular forms with level > 1 and proved it when the form is a Yoshida lift.

Suppose we are given a paramodular form $F \in S^k(\Gamma^{\text{para}}[p])$ so that for all $n \in \mathbb{Z}$, $F|T(n) = \lambda_{F,n}F = \lambda_n F$ where $T(n)$ is the nth Hecke operator. Then we can define the spin L-series by the Euler product

$$L(F,s) := \prod_{q \text{ prime}} L_q(q^{-s-k+3/2})^{-1},$$

where the local Euler factors are given by

$$L_q(X) := 1 - \lambda_q X + (\lambda_q^2 - \lambda_{q^2} - q^{2k-4})X^2 - \lambda_q q^{2k-3} X^3 + q^{4k-6} X^4$$

for $q \neq p$, and $L_p(X)$ has a similar formula.

We define

$$A_F(D) := \sum_{\{T>0 \,:\, \mathrm{disc}\, T = D\}/\hat{\Gamma}_0(p)} \frac{a(T;F)}{\varepsilon(T)}$$

where $\varepsilon(T) := \#\{U \in \hat{\Gamma}_0(p) : T[U] = T\}$.

Conjecture (Paramodular Böcherer's Conjecture, I)

Suppose $F \in S^k(\Gamma^{\mathrm{para}}[p])^+$. Then, for fundamental discriminants $D < 0$ we have

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

where C_F is a positive constant that depends only on F, and $\star = 1$ when $p \nmid D$, and $\star = 2$ when $p \mid D$.

Theorem (R., Tornaría)

Let $F = Grit(f) \in S^k(\Gamma^{para}[p])^+$ *where* p *is prime and* f *is a Hecke eigenform of degree 1, level* p *and weight* $2k-2$*. Then there exists a constant* $C_F > 0$ *so that*

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

for $D < 0$ *a fundamental discriminant, and* $\star = 1$ *when* $p \nmid D$*, and* $\star = 2$ *when* $p \mid D$*.*

The idea of the proof is to combine four ingredients:

- the factorization of the L-function of the Gritsenko lift as given by Ralf Schmidt,
- Dirichlet's class number formula,
- the explicit description of the Fourier coefficients of the Gritsenko lift and
- Waldspurger's theorem.

Theorem (R., Tornaría)

Let $F \in S^2(\Gamma^{para}[p])^+$ where $p < 600$ is prime. Then, numerically, there exists a constant $C_F > 0$ so that

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

for $-200 \leq D < 0$ a fundamental discriminant, and $\star = 1$ when $p \nmid D$, and $\star = 2$ when $p \mid D$.

Results of Cris Poor and Dave Yuen:

- Determine what levels of weight 2 paramodular cuspforms have Hecke eigenforms that are not Gritsenko lifts.
- Provide Fourier coefficients (up to discriminant 2500) for all paramodular forms of prime level up to 600 that are not Gritsenko – not enough to compute central values of twists.

Brumer and Kramer formulated the following conjecture:

Conjecture (Paramodular Conjecture)

Let p be a prime. There is a bijection between lines of Hecke eigenforms $F \in S^2(\Gamma^{\mathrm{para}}[p])$ that have rational eigenvalues and are not Gritsenko lifts and isogeny classes of rational abelian surfaces $\mathcal{A}$ of conductor p. In this correspondence we have that

$$L(\mathcal{A}, s, \text{Hasse-Weil}) = L(F, s).$$

We remark that it is merely expected that the two L-series mentioned above have an analytic continuation and satisfy a functional equation.

In our computations we assume the Paramodular conjecture for these curves:

p	ϵ	C
277	$+$	$y^2 + y = x^5 - 2x^3 + 2x^2 - x$
349	$+$	$y^2 + y = -x^5 - 2x^4 - x^3 + x^2 + x$
389	$+$	$y^2 + xy = -x^5 - 3x^4 - 4x^3 - 3x^2 - x$
461	$+$	$y^2 + y = -2x^6 + 3x^5 - 3x^3 + x$
523	$+$	$y^2 + xy = -x^5 + 4x^4 - 5x^3 + x^2 + x$
587	$+$	$y^2 = -3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$
587	-	$y^2 + \left(x^3 + x + 1\right) y = -x^3 - x^2$

The Selberg data we use are:

- $L^*(F,s) = \left(\frac{\sqrt{p}}{4\pi^2}\right)^s \Gamma(s+1/2)\Gamma(s+1/2)L(F,s)$.
- conjecturally $L^*(F,s) = \epsilon\, L^*(F,1-s)$ when $F \in S^2(\Gamma^{\mathrm{para}}[p])^\epsilon$.
- we use Mike Rubinstein's `lcalc` to compute the central values using this Selberg data and Sage code we wrote to compute the coefficients of the Hasse-Weil L-function

D	$A(D; F_{277})$	$\frac{L(F_{277},1/2,\chi_D)}{C_{277}}\,\lvert D\rvert$	D	$A(D; F_{277})$	$\frac{L(F_{277},1/2,\chi_D)}{C_{277}}\,\lvert D\rvert$
-3	-1	1.000000	-83	6	36.000000
-4	-1	1.000000	-84	1	1.000000
-7	-1	1.000000	-87	-3	9.000000
-19	-2	4.000000	-88	-2	4.000000
-23	0	-0.000000	-91	-1	1.000000
-39	1	1.000000	-116	3	9.000000
-40	-6	36.000000	-120	-2	4.000000
-47	0	0.000000	-123	-1	1.000000
-52	5	25.000000	-131	-10	100.000000
-55	-2	4.000000	-136	-6	36.000000
-59	3	9.000000	-155	-10	100.000000
-67	-8	64.000000	-164	-5	25.000000
-71	2	4.000000	-187	8	64.000001
-79	0	0.000000	-191	2	3.999999

Two surprises

Suppose $F \in S^k(\Gamma^{\mathrm{para}}[p])^-$, and let $D < 0$ be a fundamental discriminant.

- When $\left(\frac{D}{p}\right) = +1$, the Conjecture holds trivially. Indeed, note that for such F the sign of the functional equation is -1 and so the central critical value $L(F, s, \chi_D)$ is zero. On the other hand, $A(D)$ can be shown to be zero using the Twin map defined by Poor and Yuen.
- On the other hand, the formula of Conjecture 1 fails to hold in case $\left(\frac{D}{p}\right) = -1$. Since $A(D)$ is an empty sum for this type of discriminant, the right hand side of the formula vanishes trivially. However, the left hand side is still an interesting central value, not necessarily vanishing.

Two surprises

Let $L_D := L(F^-_{587}, 1/2, \chi_D)\,|D|$. This table shows fundamental discriminants for which $\left(\frac{D}{587}\right) = -1$. The obvious thing to notice is that the numbers in the table appear to be squares and so the natural question to ask is: squares of what?

D	-4	-7	-31	-40	-43	-47
L_D/L_{-3}	1.0	1.0	4.0	9.0	144.0	1.0

Two surprises

Up to now we have only considered twists by imaginary quadratic characters; namely $\chi_D = \left(\frac{\cdot}{D}\right)$ for $D < 0$. What if we consider positive D?

- Since

$$A(D) = A_F(D) := \frac{1}{2} \sum_{\{T>0:\mathrm{disc}\, T = D\}/\hat{\Gamma}_0(p)} \frac{a(T;F)}{\varepsilon(T)}$$

  we see that for $D > 0$ the sum is empty. And so Böcherer's Conjecture shouldn't make sense.

Two surprises

Let $L_D := L(F_{277}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{277}\right) = +1$. Again, these seem to be squares, but squares of what?

D	12	13	21	28	29	40
L_D/L_1	225.0	225.0	225.0	225.0	2025.0	900.0

A new conjecture

Conjecture

Let N be squarefree. Suppose $F \in S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}[N])$ is a Hecke eigenform and not a Gritsenko lift. Let ℓ and d be fundamental discriminants such that $\ell d < 0$ and such that ℓd is a square modulo $4N$. Then

$$B_{\ell,F}(\ell d)^2 = k_F \cdot \left\{2^{\nu_N(\ell)}\, L(F,1/2,\chi_\ell)\, |\ell|^{k-1}\right\} \cdot \left\{2^{\nu_N(d)}\, L(F,1/2,\chi_d)\, |d|^{k-1}\right\}$$

for some positive constant k_F independent of ℓ and d.

A new conjecture

Fix ρ such that $\rho^2 \equiv \ell d \pmod{4N}$. Then

$$B_{\ell,F}(\ell d) = \left| 2^{\nu_N(\gcd(\ell,d))} \cdot \sum \psi_\ell(T) \frac{a(T;F)}{\varepsilon(T)} \right|$$

where the sum is over $\{T = [Nm, r, n] > 0 : \operatorname{disc} T = \ell d,\ r \equiv \rho \pmod{2N}\}/\Gamma_0(N)$ and where $\psi_\ell(T)$ is the genus character corresponding to $\ell \mid \operatorname{disc} T$. This is independent of the choice of ρ.

- Essentially, $B_{\ell,F}(\ell d)$ is the same sum as $A_F(\ell d)$, but appropriately twisted by the genus character ψ_ℓ.

Proposition

Let $d > 1$ and assume Böcherer's Conjecture. Then, the ratio of special values $2^{\nu_{277}(d)} L(F_{277}, 1/2, \chi_d) |d| / L(F_{277}, 1/2)$ is divisible by 15^2.

- We note that the torsion of C_{277} is 15.
- We conjecture that this result generalizes to the other forms we considered (the data back this up) and even more generally.

Computational challenges

1. Computing enough Fourier coefficients:
   - to get Hecke eigenvalues
   - to use Böcherer's conjecture to do statistics on Siegel modular form L-functions
2. Computing enough Hecke eigenvalues:
   - to get Euler factors
   - to check cogruences
   - to use Faltings-Serre

Workaround I: New way to compute Hecke eigenvalues

- Joint with Ghitza, based on an idea of Voight.
- The action of Hecke is defined as follows: take a double coset $\Gamma M \Gamma$ and decompose it as $\Gamma M \Gamma = \cup \Gamma \alpha$. Then

$$(F|_k \Gamma M \Gamma)(Z) = \sum (F|_k \alpha)(Z)$$
$$= \sum F((AZ+B)(CZ+D)^{-1}) \text{ where } \alpha = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right).$$

- Compute eigenvalues this way! Fix a Z in the upper half-space. If F is an eigenform, compute $F(Z)$ and the $F((AZ+B)(CZ+D)^{-1})$ above. The quotient should be the eigenvalue.
- Based on work of Bröker and Lauter in which they explain how to evaluate Siegel modular forms with rigorous error bounds.

Workaround II: New way to compute a lot of Fourier coefficients

- Joint with Rupert, Sirolli and Tornaría.
- Identify as many of the first Fourier Jacobi coefficients of the form we want to compute as possible using existing data. Identify the Jacobi forms using the modular symbols method to compute Jacobi forms. Use existing techniques to compute a large of coefficients of those Jacobi forms.
- Bootstrap from here by using relations between the Fourier coefficients of Siegel forms and relations between the Fourier Jacboi coefficients of Siegel forms.

Workaround III: Evaluating L-functions with few coefficients

- Joint with Farmer.
- Using the approximate functional equation we can vary the test function that We use to evaluate $L(s)$ for a fixed s.
- we find an optimal test function by finding the least squares fit to minimize The error based on assuming the Ramanujan conjecture.
- using our method we were able to evaluate the degree 10 L-function associated to a Siegel modular form of weight 20 at $s = \frac{1}{2} + 5i$ to an error of ± 0.00016.
- This is only using 79 Euler factors!
- Computing things naively, we got the error was bigger than the value.