Explicit methods in the theory of Jacobi forms of lattice index and over number fields

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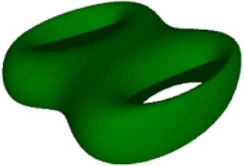

Jacobi forms over $\mathbb{Q}$: Basic example. I

Jacobi's theta function

$$\vartheta(\tau, z) = \sum_{r\in\mathbb{Z}} \left(\frac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}}$$

$$= q^{\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{n>0} \left(1 - q^n\right)\left(1 - q^n\zeta\right)\left(1 - q^n\zeta^{-1}\right)$$

Notation

$q = e^{2\pi i\tau}$, $\zeta = e^{2\pi iz}$ for $\tau \in \mathbb{H}$, $z \in \mathbb{C}$

Theorem

$\vartheta \in J_{\frac{1}{2},\frac{1}{2}}(\varepsilon^3)$.

Jacobi forms over $\mathbb{Q}$: Basic example. II

Automorphic Properties of $\vartheta(\tau, z)$

For all $A = \left[\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right]$ in $\mathrm{SL}(2,\mathbb{Z})$ and all integers λ, μ:

- $\vartheta\big(A\tau, \frac{z}{c\tau+d}\big)\, e\Big(\frac{-c\frac{1}{2}z^2}{c\tau+d}\Big)\,(c\tau+d)^{\frac{1}{2}} = \zeta_8(A)\,\vartheta(\tau, z)$
- $\vartheta\big(\tau, z+\lambda\tau+\mu\big)\, e\Big(\tau\frac{1}{2}\lambda^2 + 2\lambda\frac{1}{2}z\Big) = e\Big(\frac{1}{2}(\lambda+\mu)^2\Big)\,\vartheta(\tau, z)$

- $\frac{1}{2}$ = weight,
- $\frac{1}{2}$ = index

Jacobi forms over $\mathbb{Q}$ I

Theorem (Zagier-S.)

For every $m > 0$ and (integral) $k \geq 2$, there exist Hecke-equivariant isomorphisms

$$J_{k,m} \xrightarrow{\cong} \mathfrak{M}^{-}_{2k-2}(m),$$

where $\mathfrak{M}^{-}_{2k-2}(m)$ is a certain subspace of $M_{2k-2}\big(\Gamma_0(m)\big)$, containing all newforms whose L-series have a minus sign in their functional equation.

Jacobi forms over $\mathbb{Q}$ II

Let f be a newform in $\mathfrak{M}_{2k-2}^{-}(m)$, let

$$\phi = \sum_{4mn - r^2 \geq 0} c_\phi(n,r) q^n \zeta^r$$

be its associated Jacobi form.

Theorem (Waldpurger,Gross-Kohnen-Zagier)

For every negative fundamental discriminant $D \equiv \square \bmod 4m$, say, $D = r^2 - 4mn$, one has

$$\frac{|c_\phi(n,r)|^2}{|\phi|^2} = \mathrm{const}(k,m,D)\, \frac{L(f \otimes \chi_D, k-1)}{|f|^2}.$$

Computation of Jacobi forms over $\mathbb{Q}$

Generating explicit formulas for Jacobi forms is as easy as for elliptic modular forms (over congruence subgroups) — in fact, easier.

Methods for generating Jacobi forms

1. Theta blocks,
2. Taylor expansion around $z = 0$,
3. Modular symbols,
4. Vector valued modular forms.
5. Invariants of Weil representations and pullback.

Computation via theta blocks

- The *first* elliptic curve over $\mathbb{Q}$ of positive rank is
$$E : y^2 + y = x^3 - x \qquad \text{conductor} = 37.$$

- The associated newform f (so that $L(E,s) = L(f,s)$) is
$$f_E = q - 2q^2 - 3q^3 + 2q^4 - 2q^5 + 6q^6 - q^7 + 6q^9 + O(q^{10}).$$

- The associated Jacobi form is a theta block:
$$\phi_E = \vartheta_a \, \vartheta_b \, \vartheta_c \, \vartheta_d \, \vartheta_{a+b} \, \vartheta_{b+c} \, \vartheta_{c+d} \, \vartheta_{a+b+c} \, \vartheta_{b+c+d} \, \vartheta_{a+b+c+d} / \eta^6,$$
where $(a,b,c,d) = (1,1,1,2)$, and where $\vartheta_a(\tau,z) = \vartheta(\tau, az)$.

Recall

$$\vartheta(\tau,z) = \sum_{r\in\mathbb{Z}} \left(\tfrac{-4}{r}\right) q^{\frac{r^2}{8}} \zeta^{\frac{r}{2}} = q^{\frac{1}{8}} \left(\zeta^{\frac{1}{2}} - \zeta^{-\frac{1}{2}}\right) \prod_{n>0} \left(1-q^n\right)\left(1-q^n\zeta\right)\left(1-q^n\zeta^{-1}\right).$$

Computation via periods

Example

- The elliptic curve of congruent numbers: $C : y^2 = x(x - D)(x + D)$.
- Associated Jacobi form ϕ_C is in (spans) $J_{2,32}^{\mathrm{cusp},+}$,
- $c_\phi(n, r) = \nu_+(r^2 - 128n, r) - \nu_-(r^2 - 128n, r)$,
- For $D > 0$, $D \equiv r^2 \bmod 128$:

$$\nu_\epsilon(D, r) = \#\Big\{(a, b, c) \in \mathbb{Z}^3 \colon b^2 - 4ac = D,\ b^2 < D,\ \epsilon a > 0,\\ a \equiv 3\frac{b + r}{2} \bmod 32,\ 3c \equiv \frac{b - r}{2} \bmod 32\Big\}.$$

Remark

Slightly cheated: we used *skew-holomorphic* Jacobi forms.

A first long-term project

Goal

Develop a similar theory for Jacobi forms of several z-variables (which we shall call "Jacobi forms of lattice index").

Motivation

- Seemingly complicated Jacobi forms are pullbacks of simple universal Jacobi forms of several variables (e.g. the $m = 37$ example and infinitely many others.)
- This yields a unified arithmetic theory for all kind of (elliptic) vector valued modular forms, namely:

Theorem (S. 2012)

Any given space of elliptic modular forms of vector valued elliptic modular forms of integral or half integral weight on a congruence subgroup can be naturally embedded into a space of Jacobi forms of integral weight on the full modular group.

Definitions, notations

Definition

(Even integral positive) Lattice $\underline{L} = (L, \beta)$:

- Finite free $\mathbb{Z}$-module L,
- symmetric, positive definite $\mathbb{Z}$-bilinear map $\beta : L \times L \to \mathbb{Z}$ and

$$\beta(x) := \frac{1}{2}\beta(x, x) \text{ integral.}$$

Definition

$J_{k,\underline{L}}$ (k integral): space of holomorphic $\phi(\tau, z)$ ($\tau \in \mathbb{H}$, $z \in \mathbb{C} \otimes_{\mathbb{Z}} L$) such that:

- $\phi(\tau + 1, z) = \phi(-1/\tau, z/\tau)\, \mathrm{e}\big(-\beta(z)/\tau\big)\tau^{-k} = \phi(\tau, z)$,
- $\phi(\tau, z + \tau x + y)\, \mathrm{e}\big(\tau\beta(x) + \beta(z, x)\big) = \phi(\tau, z)$ $(x, y \in L)$,
- ϕ holomorphic at infinity.

Remarks

Fourier expansion

ϕ is called holomorphic at infinity if its Fourier expansion is of the form

$$\phi = \sum_{\substack{n \in \mathbb{Z},\ r \in L^\sharp \\ n \geq \beta(r)}} c_\phi(n, r)\, q^n\, \mathrm{e}\big(\beta(r, z)\big).$$

$(L^\sharp = \{y \in \mathbb{Q} \otimes_\mathbb{Z} L : \beta(y, L) \subseteq \mathbb{Z}\})$

Proposition

For fixed $D \leq 0$, the map $C_\phi(D, r) := c_\phi(\beta(r) - D, r)$ for $D \equiv \beta(r) \bmod \mathbb{Z}$, and $C_\phi(D, r) := 0$ otherwise, depends only on $r + L$.

Remark

Let $\underline{\mathbb{Z}}(2m) := (\mathbb{Z}, (x, y) \mapsto 2mxy)$. Then $J_{k,\underline{\mathbb{Z}}(2m)}$ equals "classical" $J_{k,m}$.

Examples

A simple effective construction method

Let $\alpha = (\alpha_1, \dots, \alpha_m)$ be an isometric embedding of $\underline{L}$ into $\underline{\mathbb{Z}}^m$. Then, for sufficiently large (possibly negative) $t \equiv -3m \bmod 24$, the function $\vartheta(\tau, \alpha_1(z)) \cdots \vartheta(\tau, \alpha_m(z))\, \eta^t$ defines an element of $J_{k,\underline{L}}$.
(If z_j are coordinate functions with respect to a $\mathbb{Z}$-basis of L, the $\alpha_j(z)$ become linear forms in z_j with integral coefficients.)

Examples

- $\vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 + z_2)\eta^{15} \in J_{9,\mathrm{A}_2}$,
- $\vartheta(\tau, z_1)\vartheta(\tau, z_2)^3\vartheta(\tau, z_1 + z_2)\eta^9 \in J_{7,\text{"}\left[\begin{smallmatrix} 2 & 1 \\ 1 & 4 \end{smallmatrix}\right]\text{"}}$,
- $\vartheta(\tau, z_1)\vartheta(\tau, z_2)\vartheta(\tau, z_1 - z_2)\vartheta(\tau, z_1 + z_2)\vartheta(\tau, z_1 + 2z_2)\vartheta(\tau, 2z_1 + z_2)/\eta(\tau)^4 \in J_{1,\text{"}\left[\begin{smallmatrix} 8 & 4 \\ 4 & 8 \end{smallmatrix}\right]\text{"}}$.

Basic features of the theory

What is known

- dim $J_{k,\underline{L}}$ = explicit formula (for <u>all</u> k, including singular or critical)
- $\bigoplus_{k\in\mathbb{Z}} J_{k,\underline{L}}$ is finite free $\mathbb{C}[E_4, E_6]$-module with explicit Hilber-Poincaré series $\frac{p(x)}{(1-x^4)(1-x^6)}$. ($p$ polynomial of weight < 12, coefficients give number of generators in a given weight.)
- Various methods for generating explicit closed formulas for Jacobi forms:
  1. $\otimes$-products ($\approx$ orthogonal sums of lattices),
  2. pull-backs ($\approx$ isometric maps between lattices),
  3. Taylor expansion in around $z = 0$ yields $J_{k,\underline{L}}$ as finite direct sum of spaces of quasi-modular forms,
  4. Forms of *singular* ($k = \frac{n}{2}$) and *critical weight* ($k = \frac{n+1}{2}$) are in 1–1 correspondence with invariants of Weil representations.

Hecke operators: odd rank

Theorem (Ajouz-S. 2015)

Let $\underline{L}$ be a lattice of odd rank n, level N, discriminant $\Delta = (-1)^{\frac{n-1}{2}} 2\det(\underline{L})$, and let ϕ be a Jacobi form in $J_{k,\underline{L}}$. For a positive integer ℓ, relatively prime to N, let

$$T(\ell)\phi := \sum_{\substack{D \le 0,\ r \in L^\sharp \\ D \equiv \beta(r) \bmod \mathbb{Z}}} C_{T(\ell)}(D,r)\, \mathrm{e}\big((\beta(r)-D)\tau + \beta(r,z)\big),$$

where

$$C_{T(\ell)\phi}(D,r) = \sum_a a^{2k-n-1}\, \rho(D,a)\, C_\phi\big(\tfrac{\ell^2}{a^2} D, \ell a' r\big).$$

Here a is over all $a | \ell^2$, $a^2 | \ell^2 ND$, $a'a \equiv 1 \bmod N$, and $\rho(D,a)$ equals $\left(\frac{D\Delta/f^2}{a/f^2}\right)$ if $\gcd(ND,a) = f^2$, and it equals 0 otherwise.
The application $\phi \mapsto T(\ell)\phi$ defines an endomorphism of $J_{k,\underline{L}}$.

Hecke operators: even rank

Theorem (Ajouz-S. 2015)

Let $\underline{L}$ be a lattice of <u>even</u> rank n, level N, discriminant $\Delta = (-1)^{\frac{n}{2}} \det(\underline{L})$, and let ϕ be a Jacobi form in $J_{k,\underline{L}}$. For a positive integer ℓ, relatively prime to N, let

$$T(\ell)\phi := \sum_{\substack{D \leq 0,\ r \in L^\sharp \\ D \equiv \beta(r) \bmod \mathbb{Z}}} C_{T(\ell)}(D, r)\, \mathrm{e}\big((\beta(r) - D)\tau + \beta(r, z)\big),$$

where

$$C_{T(\ell)\phi}(D, r) = \sum_{a | \ell^2,\ ND} a^{k-n/2} \left(\frac{\Delta}{a}\right) C_\phi\big(\tfrac{\ell^2}{a^2} D, \ell a' r\big).$$

The application $\phi \mapsto T(\ell)\phi$ defines an endomorphism of $J_{k,\underline{L}}$.

L-series

Theorem

$J_{k,\underline{L}}$ possesses a basis of simultaneous Hecke eigenforms for all ℓ (with $\gcd(\ell, N) = 1$).

Theorem

Let ϕ be a simultaneous Hecke eigenform with eigenvalues $\lambda(\ell)$, and let $L(\phi, s) = \sum_{\gcd(\ell,N)=1} \lambda(\ell)\ell^{-s}$. Then one has, for <u>odd</u> rank n,

$$L(\phi, s) = \prod_{p \nmid N} (1 - \lambda(p)p^{-s} + p^{2k-n-2-2s})^{-1},$$

and, for <u>even</u> rank n, with $\lambda'(p) = \lambda(p) - p^{k-n/2-1} \left(\frac{\Delta}{p}\right)$

$$L(\phi, s) = \frac{L(\left(\frac{\Delta}{\cdot}\right), s - k + n/2 + 1)}{\zeta^{(N)}(2s - 2k + n + 2)} \prod_{p \nmid N} (1 - \lambda'(p)p^{-s} + p^{2(k-n/2-1-2s)})^{-1}.$$

Consequences: odd rank

Conjecture

For each $k > \frac{n+1}{2}$ and each $\underline{L}$, there are Hecke equivariant injections

$$J_{k,\underline{L}} \to M_{2k-n-1}(N/4).$$

Remark

- The conjecture is true if $\underline{L}$ is stably isomorphic to a rank 1 lattice.
- The conjecture is true for Eisenstein series.
- The conjecture is true for many examples.

Expectation

More (new) finite closed formulas for $L(f \otimes \left(\frac{D}{\cdot}\right), k - \frac{n-1}{2})$ for Hecke eigenforms f in $M_{2k-n-1}(N/4)$, in particular, for f with $+1$ in functional equation.

Consequences: even rank

Observation

For even rank, the shape of the L-series of a Hecke eigenform ϕ is like the L-series $\sum \overline{\xi(\ell)}\gamma(\ell^2)\ell^{-s}$, where $\gamma(\ell)$ are the eigenvalues of a Hecke eigenform in $M_{k-n/2}(N, \xi\left(\frac{\Delta}{\cdot}\right))$ for some ξ.

Conjecture

For each $k > \frac{n}{2}$ and each $\underline{L}$, there are maps $M_{k-n/2}(N, \xi\left(\frac{\Delta}{\cdot}\right)) \to J_{k,\underline{L}}$ such that $T(\ell^2)$ on the left corresponds to $\xi(\ell)T(\ell)$ on the right. The space $J_{k,\underline{L}}$ is the sum of the images of all these maps

Remark

- The conjecture is true if $\det(\underline{L})$ is a prime.
- The conjecture is true for Eisenstein series.
- The conjecture is true for many examples.

A second long term project

Goal

Develop a similar theory over (totally real) number fields.

Problems to be solved (joint with Boylan, Hayashida, Strömberg)

- Correct definition of Jacobi forms over number fields. ✓✓
- Answer various natural questions concerning Hilbert modular groups: Non-trivial central twofold extensions, Weil representations, linear characters, ... ✓✓
  but ... $\mathrm{H}^2(\mathrm{SL}(2,\mathfrak{o}),\{\pm 1\}) = 2^?$, $[\mathrm{SL}(2,\mathfrak{o}), \prod_{v|\infty} \kappa_v]$ does not split?
- Hecke theory. ?
- Kernel functions, dimension formulas, trace formulas. ✓
- Generate examples of Jacobi forms over number fields. ✓
- Develop algorithms to generate such examples systematically. ✓

The correct definition

Remarks

1. The definitions of Jacobi forms over number fields found in the literature were incomplete.
2. In the literature there were no examples - except for some Jacobi theta series associated to lattices (H. Stark, O. Richter, ...).

For the correct definition

- Recall: The index of a Jacobi form is actually the Gram matrix of the lattice $(\mathbb{Z}, (x, y) \mapsto 2mxy)$.
- A lattice over a number field K is in general not free, and thus has no Gram matrix.
- In general one has to consider JFs over $\mathrm{SL}(\mathfrak{g} \oplus \mathfrak{o}) = \begin{bmatrix} \mathfrak{o} & \mathfrak{g}^{-1} \\ \mathfrak{g} & \mathfrak{o} \end{bmatrix} \cap \mathrm{SL}(2, K)$, where $\mathfrak{g}$ is an ideal and $\mathfrak{o}$ the ring of integers of K.

Examples: The forms of singular weight

Theorem (H. Boylan)

Let $\mathfrak{a}$ be a fractional $\mathfrak{o}$-ideal and ω a totally positive element in K such that $2\mathfrak{m}(=\mathfrak{a}^2\omega\mathfrak{d}) = \mathfrak{g}$ for a product $\mathfrak{g}$ of primes of degree 1 over 3. Suppose 2 splits completely in K. Set

$$\vartheta_{[\mathfrak{a},\omega]}(\tau, z) := \sum_{s\in\mathfrak{a}\mathfrak{g}^{-1}} \chi_{4\mathfrak{g}}(s')q^{\frac{1}{8}\omega s^2}\zeta^{\omega s/2}.$$

Then $\vartheta_{[\mathfrak{a},\omega]}$ is a Jacobi form on the full modular group of weight $1/2$, index $[\mathfrak{a},\omega] := \big(\mathfrak{a}, (x,y)\mapsto \omega xy\big)$ (with a certain character $\varepsilon_{[\mathfrak{a},\omega]}$). Here $\chi_{4\mathfrak{g}}$ is the totally odd Dirichlet character modulo $4\mathfrak{g}$, and $s\mapsto s'$ is an isomorphism of $\mathfrak{o}$-modules $\mathfrak{a}\mathfrak{g}^{-1}/4\mathfrak{a} \to \mathfrak{o}/4\mathfrak{g}$. and

Theorem (H. Boylan)

There are no other Jacobi forms of singular weight then the ones above.

A glimpse on a beautiful theory

Construction method: Taylor expansion around 0

- A classical Jacobi form $\phi(\tau, z)$ of index m has, for fixed τ as function of z, exactly $2m$ zeros in $\mathbb{C}/\mathbb{Z}\tau + \mathbb{Z}$.
- Therefore, writing $\phi(\tau, z) = \sum_{n \geq 0} \alpha_n(\tau)\, z^n$, the first m many $\alpha_k(\tau)$ determine ϕ.
- The $\alpha_j(\tau)$ are quasi-modular forms (i.e. elements of $\mathbb{C}[E_2, E_4, R_6]$).
- Leads to *useful* description of Jacobi forms in terms of quasi-modular forms.

Somehow surprising fact

This holds true for Jacobi forms over number fields too.

Consequence

For given lattice over a number field the Jacobi forms of this lattice index (and arbitrary weight) form a *finitely generated* module over the ring of Hilbert modular forms on the full Hilbert modular group.

References

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Thank you!