Non-archimedean construction of elliptic curves and abelian surfaces

ICERM WORKSHOP
Modular Forms and Curves of Low Genus: Computational Aspects

Xavier Guitart [1] Marc Masdeu [2] Mehmet Haluk Sengun [3]

[1]Universitat de Barcelona

[2]University of Warwick

[3]University of Sheffield

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Quaternionic automorphic forms of level $\mathfrak{N}$

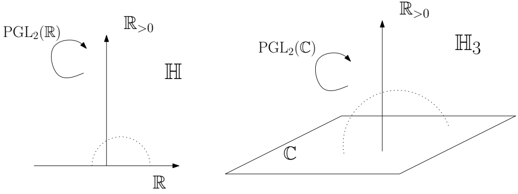
- F a number field of signature (r, s), and fix $\mathfrak{N} \subset \mathcal{O}_F$.
- Choose factorization $\mathfrak{N} = \mathfrak{D}\mathfrak{n}$, with $\mathfrak{D}$ **square free.**
- Fix embeddings $v_1, \ldots, v_r \colon F \hookrightarrow \mathbb{R}, \quad w_1, \ldots, w_s \colon F \hookrightarrow \mathbb{C}$.
- Let B/F be a quaternion algebra such that

$$\mathrm{Ram}(B) = \{\mathfrak{q} \colon \mathfrak{q} \mid \mathfrak{D}\} \cup \{v_{n+1}, \ldots, v_r\}, \quad (n \leqslant r).$$

- Fix isomorphisms

$$B \otimes F_{v_i} \cong M_2(\mathbb{R}), \quad i = 1, \ldots, n; \quad B \otimes F_{w_j} \cong M_2(\mathbb{C}), \quad j = 1, \ldots, s.$$

- These yield $B^\times/F^\times \hookrightarrow \mathrm{PGL}_2(\mathbb{R})^n \times \mathrm{PGL}_2(\mathbb{C})^s \circlearrowright \mathbb{H}^n \times \mathbb{H}_3^s$.



Quaternionic automorphic forms of level $\mathfrak{N}$ (II)

- Fix $R_0^{\mathfrak{D}}(\mathfrak{n}) \subset B$ Eichler order of level $\mathfrak{n}$.
- $\Gamma_0^{\mathfrak{D}}(\mathfrak{n}) = R_0^{\mathfrak{D}}(\mathfrak{n})^\times / \mathcal{O}_F^\times$ acts **discretely** on $\mathbb{H}^n \times \mathbb{H}_3^s$.
- Obtain an orbifold of (real) dimension $2n + 3s$:

$$Y_0^{\mathfrak{D}}(\mathfrak{n}) = \Gamma_0^{\mathfrak{D}}(\mathfrak{n}) \backslash \left(\mathbb{H}^n \times \mathbb{H}_3^s\right).$$

- The cohomology of $Y_0^{\mathfrak{D}}(\mathfrak{n})$ can be computed via

$$H^*(Y_0^{\mathfrak{D}}(\mathfrak{n}), \mathbb{C}) \cong H^*(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}), \mathbb{C}).$$

- Hecke algebra $\mathbb{T}^{\mathfrak{D}} = \mathbb{Z}[T_{\mathfrak{q}} \colon \mathfrak{q} \nmid \mathfrak{D}]$ acts on $H^*(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}), \mathbb{Z})$.

$$H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}), \mathbb{C}) = \bigoplus_{\chi} H^{n+s}(\Gamma_0^{\mathfrak{N}}(\mathfrak{n}), \mathbb{C})^{\chi}, \quad \chi \colon \mathbb{T}^{\mathfrak{D}} \to \mathbb{C}.$$

Each χ cuts out a field K_χ, s.t. $[K_\chi \colon \mathbb{Q}] = \dim H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}), \mathbb{C})^{\chi}$.

Abelian varieties from cohomology classes

Definition

$f \in H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}),\mathbb{C})^{\chi}$ eigen for $\mathbb{T}^{\mathfrak{D}}$ is **rational** if $a_{\mathfrak{p}}(f) \in \mathbb{Z}, \forall \mathfrak{p} \in \mathbb{T}^{\mathfrak{D}}$.

- If $r = 0$, then **assume** $\mathfrak{N}$ is not square-full: $\exists \mathfrak{p} \parallel \mathfrak{N}$.

Conjecture (Taylor, ICM 1994)

1. *$f \in H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{n}),\mathbb{Z})$ a **new, rational eigenclass**.*
   Then $\exists E_f/F$ of conductor $\mathfrak{N} = \mathfrak{D}\mathfrak{n}$ attached to f. i.e. such that

$$\#E_f(\mathcal{O}_F/\mathfrak{p}) = 1 + |\mathfrak{p}| - a_{\mathfrak{p}}(f) \quad \forall \mathfrak{p} \nmid \mathfrak{N}.$$

2. *More generally, if $\chi\colon \mathbb{T}^{\mathfrak{D}} \to \mathbb{C}$ is nontrivial, cutting out a field K, then $\exists$ abelian variety A_χ, with $\dim A_\chi = [K\colon F]$ and multiplication by K.*

- Assumption above avoids “fake abelian varieties”, and it is needed in our construction anyway.

Goals of this talk

In this talk we will:

1. Review known **explicit** forms of this conjecture.
   - Cremona's algorithm for $F = \mathbb{Q}$.
   - Generalizations to totally real fields.
2. Propose a new, **non-archimedean**, conjectural construction.
   - (joint work with X. Guitart and H. Sengun)
3. Explain some **computational** details.
4. Illustrate with **examples.**

$F = \mathbb{Q}$: Cremona's algorithm for elliptic curves

Eichler–Shimura construction

$$X_0(N) \longrightarrow \mathrm{Jac}(X_0(N)) \overset{\int}{\cong} \frac{H^0\big(X_0(N),\Omega^1\big)^\vee}{H_1(X_0(N),\mathbb{Z})} \xrightarrow{\text{Hecke}} \mathbb{C}/\Lambda_f \cong E_f(\mathbb{C}).$$

1. Compute $H_1(X_0(N),\mathbb{Z})$ (modular symbols).
2. Find the period lattice Λ_f by explicitly integrating

$$\Lambda_f = \left\langle \int_\gamma 2\pi i \sum_{n\geqslant 1} a_n(f) e^{2\pi i n z} : \gamma \in H_1\Big(X_0(N),\mathbb{Z}\Big) \right\rangle.$$

3. Compute $c_4(\Lambda_f), c_6(\Lambda_f) \in \mathbb{C}$ by evaluating Eistenstein series.
4. Recognize $c_4(\Lambda_f), c_6(\Lambda_f)$ as integers $\rightsquigarrow E_f : Y^2 = X^3 - \frac{c_4}{48}X - \frac{c_6}{864}$.

$F \neq \mathbb{Q}$: constructions for elliptic curves

F **totally real.** $[F : \mathbb{Q}] = n$, fix $\sigma \colon F \hookrightarrow \mathbb{R}$.

$$S_2(\Gamma_0(\mathfrak{N})) \ni f \rightsquigarrow \tilde{\omega}_f \in H^n(\Gamma_0(\mathfrak{N}), \mathbb{C}) \rightsquigarrow \Lambda_f \subseteq \mathbb{C}.$$

Conjecture (Oda, Darmon, Gartner)

$\mathbb{C}/\Lambda_f$ is isogenous to $E_f \times_F F_\sigma$.

- Known to hold (when F real quadratic) for base-change of $E/\mathbb{Q}$.
- Exploited in very restricted cases (Dembélé, Stein+7).
- Explicitly computing Λ_f is hard.
  - No quaternionic computations (except for Voight–Willis?).

F **not totally real**: no known algorithms...

Theorem

*If F is **imaginary quadratic**, the lattice Λ_f is contained in $\mathbb{R}$.*

Idea

Allow for **non-archimedean** constructions.

Non-archimedean construction

- From now on: fix $\mathfrak{p} \parallel \mathfrak{N}$.
- Denote by $\bar{F}_{\mathfrak{p}}$ = alg. closure of the $\mathfrak{p}$-completion of F.

Theorem (Tate uniformization)

There exists a rigid-analytic, Galois-equivariant isomorphism

$$\eta \colon \bar{F}_{\mathfrak{p}}^{\times}/\langle q_E \rangle \to E(\bar{F}_{\mathfrak{p}}),$$

with $q_E \in F_{\mathfrak{p}}^{\times}$ satisfying $j(E) = q_E^{-1} + 744 + 196884 q_E + \cdots$.

- Choose a coprime factorization $\mathfrak{N} = \mathfrak{p}\mathfrak{D}\mathfrak{m}$, with $\mathfrak{D} = \mathrm{disc}(B/F)$.
- Compute q_E as a replacement for Λ_f.
- Starting data: $f \in H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{m}), \mathbb{Z})^{\mathfrak{p}-\text{new}}, \quad \mathfrak{p}\mathfrak{D}\mathfrak{m} = \mathfrak{N}$.

Non-archimedean path integrals on $\mathcal{H}_\mathfrak{p}$

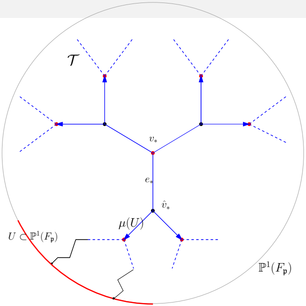
- Consider $\mathcal{H}_\mathfrak{p} = \mathbb{P}^1(\mathbb{C}_\mathfrak{p}) \smallsetminus \mathbb{P}^1(F_\mathfrak{p})$.
- It is a $\mathfrak{p}$-adic analogue to $\mathbb{H}$:
  - It has a **rigid-analytic** structure.
  - Action of $\mathrm{PGL}_2(F_\mathfrak{p})$ by **fractional linear transformations.**
  - **Rigid-analytic** 1-forms $\omega \in \Omega^1_{\mathcal{H}_\mathfrak{p}}$.
  - **Coleman integration** $\rightsquigarrow$ make sense of $\int_{\tau_1}^{\tau_2} \omega \in \mathbb{C}_\mathfrak{p}$.
- Get a $\mathrm{PGL}_2(F_\mathfrak{p})$-equivariant pairing $\int \colon \Omega^1_{\mathcal{H}_\mathfrak{p}} \times \mathrm{Div}^0\,\mathcal{H}_\mathfrak{p} \to \mathbb{C}_\mathfrak{p}$.
- For each $\Gamma \subset \mathrm{PGL}_2(F_\mathfrak{p})$, get induced pairing (cap product)

$$H^i(\Gamma, \Omega^1_{\mathcal{H}_\mathfrak{p}}) \times H_i(\Gamma, \mathrm{Div}^0\,\mathcal{H}_\mathfrak{p}) \xrightarrow{\ \int\ } \mathbb{C}_\mathfrak{p}$$

$$\left(\phi, \textstyle\sum_\gamma \underline{\gamma} \otimes D_\gamma\right) \longmapsto \textstyle\sum_\gamma \displaystyle\int_{D_\gamma} \phi(\underline{\gamma}).$$

- $\Omega^1_{\mathcal{H}_\mathfrak{p}} \cong$ space of $\mathbb{C}_\mathfrak{p}$-valued **boundary measures** $\mathrm{Meas}_0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{C}_\mathfrak{p})$.

Measures and integrals



- Bruhat-Tits tree of $\mathrm{GL}_2(F_\mathfrak{p})$, $|\mathfrak{p}| = 2$.
- $\mathbb{P}^1(F_\mathfrak{p}) \cong \mathrm{Ends}(\mathcal{T})$.
- Harmonic cocycles $\mathrm{HC}(A) = \{\mathcal{E}(\mathcal{T}) \xrightarrow{f} A \mid \sum_{o(e)=v} f(e) = 0\}$
- $\mathrm{Meas}_0(\mathbb{P}^1(F_\mathfrak{p}), A) \cong \mathrm{HC}(A)$.
- So replace $\omega \in \Omega^1_{\mathcal{H}_\mathfrak{p}}$ with $\mu_\omega \in \mathrm{Meas}_0(\mathbb{P}^1(F_\mathfrak{p}), \mathbb{Z}) \cong \mathrm{HC}(\mathbb{Z})$.
- Coleman integration: if $\tau_1, \tau_2 \in \mathcal{H}_\mathfrak{p}$, then

$$\int_{\tau_1}^{\tau_2} \omega = \int_{\mathbb{P}^1(F_\mathfrak{p})} \log_\mathfrak{p}\left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_\omega(t) = \lim_{\overrightarrow{\mathcal{U}}} \sum_{U \in \mathcal{U}} \log_\mathfrak{p}\left(\frac{t_U-\tau_2}{t_U-\tau_1}\right) \mu_\omega(U).$$

- **Multiplicative** refinement (assume $\mu_\omega(U) \in \mathbb{Z}$, $\forall U$):

$$\times\!\!\!\!\!\!\int_{\tau_1}^{\tau_2} \omega = \times\!\!\!\!\!\!\int_{\mathbb{P}^1(F_\mathfrak{p})} \left(\frac{t-\tau_2}{t-\tau_1}\right) d\mu_\omega(t) = \lim_{\overrightarrow{\mathcal{U}}} \prod_{U \in \mathcal{U}} \left(\frac{t_U-\tau_2}{t_U-\tau_1}\right)^{\mu_\omega(U)}.$$

The $\{\mathfrak{p}\}$-arithmetic group Γ

- Choose a factorization $\mathfrak{N} = \mathfrak{p}\mathfrak{D}\mathfrak{m}$.
- $B_{/F}$ = quaternion algebra with $\mathrm{Ram}(B) = \{\mathfrak{q} \mid \mathfrak{D}\} \cup \{v_{n+1}, \ldots, v_r\}$.
- Recall also $R_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \subset R_0^{\mathfrak{D}}(\mathfrak{m}) \subset B$.
- Fix $\iota_{\mathfrak{p}} \colon R_0^{\mathfrak{D}}(\mathfrak{m}) \hookrightarrow M_2(\mathbb{Z}_{\mathfrak{p}})$.
- Define $\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) = R_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})^\times/\mathcal{O}_F^\times$ and $\Gamma_0^{\mathfrak{D}}(\mathfrak{m}) = R_0^{\mathfrak{D}}(\mathfrak{m})^\times/\mathcal{O}_F^\times$.
- Let $\Gamma = R_0^{\mathfrak{D}}(\mathfrak{m})[1/\mathfrak{p}]^\times/\mathcal{O}_F[1/\mathfrak{p}]^\times \overset{\iota_{\mathfrak{p}}}{\hookrightarrow} \mathrm{PGL}_2(F_{\mathfrak{p}})$.

Example

- $F = \mathbb{Q}$ and $\mathfrak{D} = 1$, so $\mathfrak{N} = pM$.
- $B = M_2(\mathbb{Q})$.
- $\Gamma_0(pM) = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}) \colon pM \mid c \right\} / \{\pm 1\}$.
- $\Gamma = \left\{ \left(\begin{smallmatrix} a & b \\ c & d \end{smallmatrix}\right) \in \mathrm{GL}_2(\mathbb{Z}[1/p]) \colon M \mid c \right\} / \{\pm 1\} \hookrightarrow \mathrm{PGL}_2(\mathbb{Q}) \subset \mathrm{PGL}_2(\mathbb{Q}_p)$.

The $\{\mathfrak{p}\}$-arithmetic group Γ

Lemma

Assume that $h_F^+ = 1$. *Then* $\iota_{\mathfrak{p}}$ *induces bijections*

$$\Gamma/\Gamma_0^{\mathfrak{D}}(\mathfrak{m}) \cong \mathcal{V}_0(\mathcal{T}), \quad \Gamma/\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \cong \mathcal{E}_0(\mathcal{T})$$

$\mathcal{V}_0 = \mathcal{V}_0(\mathcal{T})$ *(resp.* $\mathcal{E}_0 = \mathcal{E}_0(\mathcal{T})$*) are the even vertices (resp. edges) of* $\mathcal{T}$.

Proof.

1. Strong approximation $\implies$ Γ acts transitively on $\mathcal{E}_0$ and $\mathcal{V}_0$.
2. Stabilizer of vertex v_* (resp. edge e_*) is $\Gamma_0^{\mathfrak{D}}(\mathfrak{m})$ (resp. $\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$).

□

Corollary

$$\mathrm{Maps}(\mathcal{E}_0(\mathcal{T}), \mathbb{Z}) \cong \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})}^{\Gamma} \mathbb{Z}, \quad \mathrm{Maps}(\mathcal{V}(\mathcal{T}), \mathbb{Z}) \cong \left(\mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}^{\Gamma} \mathbb{Z}\right)^2.$$

Cohomology

$$\Gamma = R_0^{\mathfrak{D}}(\mathfrak{m})[1/\mathfrak{p}]^\times/\mathcal{O}_F[1/\mathfrak{p}]^\times \stackrel{\iota_\mathfrak{p}}{\hookrightarrow} \mathrm{PGL}_2(F_\mathfrak{p}).$$

$$\mathrm{Maps}(\mathcal{E}_0(\mathcal{T}),\mathbb{Z}) \cong \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})}^{\Gamma}\mathbb{Z}, \quad \mathrm{Maps}(\mathcal{V}(\mathcal{T}),\mathbb{Z}) \cong \left(\mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}^{\Gamma}\mathbb{Z}\right)^2.$$

- Want to define a cohomology class in $H^{n+s}(\Gamma,\Omega^1_{\mathcal{H}_\mathfrak{p}})$.
- Consider the Γ-equivariant exact sequence

$$0 \longrightarrow \mathrm{HC}(\mathbb{Z}) \longrightarrow \mathrm{Maps}(\mathcal{E}^0(\mathcal{T}),\mathbb{Z}) \xrightarrow{\beta} \mathrm{Maps}(\mathcal{V}(\mathcal{T}),\mathbb{Z}) \longrightarrow 0$$
$$\varphi \longmapsto [v \mapsto \textstyle\sum_{o(e)=v}\varphi(e)]$$

- So get:

$$0 \to \mathrm{HC}(\mathbb{Z}) \to \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})}^{\Gamma}\mathbb{Z} \xrightarrow{\beta} \left(\mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}^{\Gamma}\mathbb{Z}\right)^2 \to 0$$

Cohomology (II)

$$0 \to \mathrm{HC}(\mathbb{Z}) \to \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})}^{\Gamma} \mathbb{Z} \xrightarrow{\beta} \left(\mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}^{\Gamma} \mathbb{Z}\right)^2 \to 0$$

- Taking Γ-cohomology, ...

$$H^{n+s}(\Gamma, \mathrm{HC}(\mathbb{Z})) \to H^{n+s}(\Gamma, \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})}^{\Gamma}, \mathbb{Z}) \xrightarrow{\beta} H^{n+s}(\Gamma, \mathrm{Ind}_{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}^{\Gamma}, \mathbb{Z})^2 \to \cdots$$

- ...and using Shapiro's lemma:

$$H^{n+s}(\Gamma, \mathrm{HC}(\mathbb{Z})) \to H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{Z}) \xrightarrow{\beta} H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{m}), \mathbb{Z})^2 \to \cdots$$

- $f \in H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{Z})$ being $\mathfrak{p}$-new $\Leftrightarrow f \in \mathrm{Ker}(\beta)$.

Pulling back get

$$\omega_f \in H^{n+s}(\Gamma, \mathrm{HC}(\mathbb{Z})) \cong H^{n+s}(\Gamma, \Omega^1_{\mathcal{H}_\mathfrak{p}}).$$

Holomogy

- Consider the Γ-equivariant short exact sequence:

$$0 \to \mathrm{Div}^0 \mathcal{H}_\mathfrak{p} \to \mathrm{Div}\, \mathcal{H}_\mathfrak{p} \overset{\deg}{\to} \mathbb{Z} \to 0.$$

- Taking Γ-homology yields

$$H_{n+s+1}(\Gamma, \mathbb{Z}) \overset{\delta}{\to} H_{n+s}(\Gamma, \mathrm{Div}^0 \mathcal{H}_\mathfrak{p}) \to H_{n+s}(\Gamma, \mathrm{Div}\, \mathcal{H}_\mathfrak{p}) \to H_{n+s}(\Gamma, \mathbb{Z})$$

$$\Lambda_f = \left\{ \times\!\!\!\!\!\!\int_{\delta(c)} \omega_f : c \in H_{n+s+1}(\Gamma, \mathbb{Z}) \right\} \subset \mathbb{C}_p^\times$$

Conjecture A (Greenberg, Guitart–M.–Sengun)

The multiplicative lattice Λ_f is homothetic to $q_E^\mathbb{Z}$.

- $F = \mathbb{Q}$: Darmon, Dasgupta–Greenberg, Longo–Rotger–Vigni.
- F totally real, $|\mathfrak{p}| = 1$, $B = M_2(F)$: Spiess.
- Open in general.

Lattice: explicit construction

- Start with $f \in H^{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}, \mathbb{Z}))^{\mathfrak{p}-\text{new}}$.
- Duality yields $\hat{f} \in H_{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{Z}))^{\mathfrak{p}-\text{new}}$.
- Mayer–Vietoris exact sequence for $\Gamma = \Gamma_0^{\mathfrak{D}}(\mathfrak{m}) \star_{\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})} \widehat{\Gamma_0^{\mathfrak{D}}(\mathfrak{m})}$:

$$\cdots \to H_{n+s+1}(\Gamma, \mathbb{Z}) \xrightarrow{\delta'} H_{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{Z}) \xrightarrow{\beta} H_{n+s}(\Gamma_0^{\mathfrak{D}}(\mathfrak{m}), \mathbb{Z})^2 \to \cdots$$

- $\hat{f}$ new at $\mathfrak{p}$ $\implies$ $\beta(\hat{f}) = 0$.
  - $\hat{f} = \delta'(c_f)$, for some $c_f \in H_{n+s+1}(\Gamma, \mathbb{Z})$.

Conjecture (rephrased)

The element

$$\mathcal{L}_f = \int_{\delta(c_f)} \omega_f.$$

satisfies (up to a rational multiple) $\log_{\mathfrak{p}}(q_E) = \mathcal{L}_f$.

Algorithms

- Only in the cases $n + s \leqslant 1$.
  - Both H^1 and H_1: fox calculus (linear algebra for finitely-presented groups).
- Use explicit presentation + word problem for $\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$ and $\Gamma_0^{\mathfrak{D}}(\mathfrak{m})$.
  - John Voight ($s = 0$).
  - Aurel Page ($s = 1$).
- Need the Hecke action on $H^1(\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}), \mathbb{Z})$ and $H_1(\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}), \mathbb{Z})$.
  - Shapiro's lemma $\Longrightarrow$ enough to work with $\Gamma_0^{\mathfrak{D}}(\mathfrak{m})$.
- Integration pairing uses the **overconvergent method.**

Overconvergent Method

- Starting data: cohomology class $\phi = \omega_f \in H^1(\Gamma, \Omega^1_{\mathcal{H}_\mathfrak{p}})$.
- Goal: to compute integrals $\int_{\tau_1}^{\tau_2} \phi_\gamma$, for $\gamma \in \Gamma$.
- Recall that

$$\int_{\tau_1}^{\tau_2} \phi_\gamma = \int_{\mathbb{P}^1(F_\mathfrak{p})} \log_\mathfrak{p}\left(\frac{t-\tau_1}{t-\tau_2}\right) d\mu_\gamma(t).$$

- Expand the integrand into power series and change variables.
  - We are reduced to calculating the **moments**:

$$\int_{\mathbb{Z}_\mathfrak{p}} t^i d\mu_\gamma(t) \quad \text{for all } \gamma \in \Gamma.$$

- **Note:** $\Gamma \supseteq \Gamma_0^{\mathfrak{D}}(\mathfrak{m}) \supseteq \Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m})$.
- **Technical lemma:** All these integrals can be recovered from

$$\left\{ \int_{\mathbb{Z}_\mathfrak{p}} t^i d\mu_\gamma(t) : \gamma \in \Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \right\}.$$

Overconvergent Method (II)

- $\mathbb{D} = \{$locally analytic $\mathbb{Z}_\mathfrak{p}$-valued distributions on $\mathbb{Z}_\mathfrak{p}\}$.
  - $\varphi \in \mathbb{D}$ maps a locally-analytic function h on $\mathbb{Z}_\mathfrak{p}$ to $\varphi(h) \in \mathbb{Z}_\mathfrak{p}$.
  - $\mathbb{D}$ is naturally a $\Gamma_0^{\mathfrak{D}}(\mathfrak{pm})$-module.
- The map $\varphi \mapsto \varphi(1_{\mathbb{Z}_\mathfrak{p}})$ induces a projection:

$$\begin{array}{ccc} H^1(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{D}) & \xrightarrow{\rho} & H^1(\Gamma_0^{\mathfrak{D}}(\mathfrak{pm}), \mathbb{Z}_\mathfrak{p}). \\ \Cup & & \Cup \\ \Phi & \mapsto\text{-----}\to & \phi \end{array}$$

Theorem (Pollack-Stevens, Pollack-Pollack)

There exists a unique U_p-eigenclass Φ lifting ϕ.

- Moreover, Φ is explicitly computable by iterating the U_p-operator.

Overconvergent Method (III)

But we wanted to compute the moments of a system of measures. . .

Proposition

Consider the map $\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}) \to \mathbb{D}$*:*

$$\gamma \mapsto \left[h(t) \mapsto \int_{\mathbb{Z}_{\mathfrak{p}}} h(t)d\mu_\gamma(t)\right].$$

1. *It satisfies a cocycle relation* $\implies$ *induces a class* $\Psi \in H^1\left(\Gamma_0^{\mathfrak{D}}(\mathfrak{p}\mathfrak{m}), \mathbb{D}\right)$.
2. Ψ *is a lift of* ϕ.
3. Ψ *is a* U_p*-eigenclass.*

Corollary

The explicitly computed $\Phi = \Psi$ ***knows*** *the above integrals.*

Recovering E from Λ_f

- $\Lambda_f = \langle q_f \rangle$ gives us $q_f \stackrel{?}{=} q_E$.
- Assume $\operatorname{ord}_{\mathfrak{p}}(q_f) > 0$ (otherwise, replace $q_f \mapsto 1/q_f$).
- Get

$$j(q_f) = q_f^{-1} + 744 + 196884 q_f + \cdots \in \mathbb{C}_p^\times.$$

- From $\mathfrak{N}$ guess the discriminant Δ_E.
  - Only finitely-many possibilities, $\Delta_E \in S(F, 12)$.
- $j(q_f) = c_4^3/\Delta_E \rightsquigarrow$ recover c_4.
- Recognize c_4 algebraically.
- $1728\Delta_E = c_4^3 - c_6^2 \rightsquigarrow$ recover c_6.
- Compute the conductor of $E_f \colon Y^2 = X^3 - \frac{c_4}{48}X - \frac{c_6}{864}$.
  - If conductor is correct, check $a_{\mathfrak{q}}$'s.

Example curve (joint with X. Guitart and H. Sengun)

- $F = \mathbb{Q}(\alpha), p_\alpha(x) = x^4 - x^3 + 3x - 1$, $\Delta_F = -1732$.
- $\mathfrak{N} = (\alpha - 2) = \mathfrak{p}_{13}$.
- B/F ramified only at all infinite real places of F.
- There is a rational eigenclass $f \in S_2(\Gamma_0(1, \mathfrak{N}))$.
- From f we compute $\omega_f \in H^1(\Gamma, \mathrm{HC}(\mathbb{Z}))$ and Λ_f.
- $q_f \stackrel{?}{=} q_E = 8 \cdot 13 + 11 \cdot 13^2 + 5 \cdot 13^3 + 3 \cdot 13^4 + \cdots + O(13^{100})$.
- $j_E = \frac{1}{13}\left(- 4656377430074\alpha^3 + 10862248656760\alpha^2 - 14109269950515\alpha + 4120837170980\right)$.
- $c_4 = 2698473\alpha^3 + 4422064\alpha^2 + 583165\alpha - 825127$.
- $c_6 = 20442856268\alpha^3 - 4537434352\alpha^2 - 31471481744\alpha + 10479346607$.

$$
\begin{aligned}
E_{/F} \colon y^2 + \left(\alpha^3 + \alpha + 3\right) xy = x^3 + \\
&+ \left(-2\alpha^3 + \alpha^2 - \alpha - 5\right) x^2 \\
&+ \left(-56218\alpha^3 - 92126\alpha^2 - 12149\alpha + 17192\right) x \\
&- 23593411\alpha^3 + 5300811\alpha^2 + 36382184\alpha - 12122562.
\end{aligned}
$$

Tables: Imaginary quadratic fields

$\lvert\Delta_K\rvert$	$f_K(x)$	$\mathrm{Nm}(\mathfrak{N})$	$\mathfrak{p}\mathfrak{D}\mathfrak{m}$	$c_4(E), c_6(E)$
3	$[1, -1]$	196	$(3r-2)_7(-6r+2)_{28}(1)$	$-131065r$, 47449331
3	$[1, -1]$	196	$(-3r+1)_7(6r-4)_{28}(1)$	$-131065r$, 47449331
4	$[1, 0]$	130	$(3r-2)_{13}(-r-3)_{10}(1)$	$-264r+257$, $-6580r+2583$
4	$[1, 0]$	130	$(-3r-2)_{13}(-3r-1)_{10}(1)$	$264r+257$, $6580r+2583$
7	$[2, -1]$	44	$(r)_2(3r+1)_{22}(1)$	$648r+481$, $-28836r+4447$
7	$[2, -1]$	44	$(r-1)_2(3r-4)_{22}(1)$	$-648r+1129$, $28836r-24389$
8	$[2, 0]$	99	$(r+1)_3(-4r+1)_{33}(1)$	$444r+25$, $14794r-16263$
8	$[2, 0]$	99	$(r-1)_3(-4r-1)_{33}(1)$	$-444r+25$, $-14794r-16263$
8	$[2, 0]$	99	$(-r-3)_{11}(3)_9(1)$	$-444r+25$, $-14794r-16263$
8	$[2, 0]$	99	$(r-3)_{11}(3)_9(1)$	$444r+25$, $14794r-16263$

Tables: cubic $(1,1)$ fields

$\lvert\Delta_K\rvert$	$f_K(x)$	$\mathrm{Nm}(\mathfrak{N})$	$\mathfrak{p}\mathfrak{D}\mathfrak{m}$	$c_4(E), c_6(E)$
23	$[1,0,-1]$	185	$(r^2+1)_5(3r^2-r+1)_{37}(1)$	$643318r^2-1128871r+852306$, $925824936r^2-1624710823r+1226456111$
31	$[-1,1,0]$	129	$(-r-1)_3(-3r^2-2r-1)_{43}(1)$	$-4787r^2+10585r+3349$, $12687769r^2-371369r+424764$
44	$[1,1,-1]$	121	$(2r-1)_{11}(r^2+2)_{11}(1)$	$4097022r^2-6265306r+7487000$, $14168359144r^2-21861492432r+26039140708$
44	$[1,1,-1]$	121	$(2r-1)_{11}(r^2+2)_{11}(1)$	$1774r^2-1434r-1304$, $-42728r^2-123104r-54300$
44	$[1,1,-1]$	121	$(2r-1)_{11}(r^2+2)_{11}(1)$	$4097022r^2-6265306r+7487000$, $14168359144r^2-21861492432r+26039140708$
59	$[-1,2,0]$	34	$(-r^2-1)_2(-r^2-2r-2)_{17}(1)$	$262r^2+513r+264$, $-2592r^2+448r+13231$
59	$[-1,2,0]$	34	$(-r^2-2r-2)_{17}(-r^2-1)_2(1)$	$16393r^2+20228r-12524$, $4430388r^2-5579252r+1619039$
59	$[-1,2,0]$	46	$(-2r^2+r-2)_{23}(-r^2-1)_2(1)$	$18969r^2+8532r+41788$, $4216716r^2+1911600r+9298151$
59	$[-1,2,0]$	74	$(-r^2-1)_2(2r^2+2r+1)_{37}(1)$	$33054r^2+15049r+72776$, $9702640r^2+4400116r+21401723$
59	$[-1,2,0]$	88	$(-r^2-1)_2(r-2)_{11}(r^2+r+1)_4$	$16609r^2+7084r+37332$, $3522136r^2+1613876r+7760395$
59	$[-1,2,0]$	187	$(2r^2+r+2)_{17}(r-2)_{11}(1)$	$-32r^2-848r+432$, $-7600r^2+23368r-8704$
76	$[-2,-2,0]$	117	$(2r^2-r-3)_{13}(-r^2+2r+1)_9(1)$	$48r+16$, $-128r^2-224r-216$
83	$[-2,1,-1]$	65	$(r+1)_5(-2r+1)_{13}(1)$	$3089r^2+1086r+4561$, $333604r^2+117840r+493059$
83	$[-2,1,-1]$	65	$(r+1)_5(-2r+1)_{13}(1)$	$304r^2+112r+449$, $6616r^2+2328r+9791$
83	$[-2,1,-1]$	65	$(-2r+1)_{13}(r+1)_5(1)$	$3089r^2+1086r+4561$, $333604r^2+117840r+493059$
83	$[-2,1,-1]$	65	$(-2r+1)_{13}(r+1)_5(1)$	$4499473r^2+1589254r+6650137$, $18573712184r^2+6560420272r+27451337687$
83	$[-2,1,-1]$	106	$(r)_2(2r^2-3r+3)_{53}(1)$	$2329r^2+822r+3441$, $-34264r^2-12104r-50645$
87	$[1,2,-1]$	123	$(r^2-r+1)_3(r^2+4)_{41}(1)$	$1424r^2+3792r+2384$, $-245696r^2-201800r-4144$
87	$[1,2,-1]$	129	$(r^2-r+1)_3(-3r+1)_{43}(1)$	$-752r^2+2272r+1009$, $27496r^2-152144r-63977$
87	$[1,2,-1]$	129	$(r^2-r+1)_3(-3r+1)_{43}(1)$	$-752r^2+2272r+1009$, $27496r^2-152144r-63977$
104	$[-2,-1,0]$	143	$(r^2+r-1)_{11}(2r+1)_{13}(1)$	$12r^2+12r+25$, $-144r^2-90r-125$
107	$[-2,3,-1]$	40	$(-r^2-1)_5(r^2-r+3)_4(r)_2$	$3880r^2-984r+10473$, $405820r^2-105348r+1142075$
107	$[-2,3,-1]$	135	$(-r^2-1)_5(3)_{27}(1)$	$16r^2-16r$, $184r-296$
108	$[-2,0,0]$	34	$(2r+1)_{17}(r)_2(1)$	$184r^2+212r+265$, $-5010r^2-6306r-7773$
108	$[-2,0,0]$	85	$(-r^2-1)_5(2r+1)_{17}(1)$	$2224r^2+2816r+3520$, $-229056r^2-288672r-363768$
108	$[-2,0,0]$	85	$(2r+1)_{17}(-r^2-1)_5(1)$	$2224r^2+2816r+3520$, $-229056r^2-288672r-363768$
108	$[-2,0,0]$	125	$(-r^2-1)_5(r^2-2r-1)_{25}(1)$	$496r^2$, 22088
108	$[-2,0,0]$	145	$(-r^2-1)_5(r+3)_{29}(1)$	$144r^2+176r+240$, $3816r^2+4752r+6088$
108	$[-2,0,0]$	155	$(-r^2-1)_5(r^2+3)_{31}(1)$	$16r^2+20r+17$, $-606r^2-762r-929$
116	$[-2,0,-1]$	34	$(-2r+1)_{17}(-r+1)_2(1)$	$846760r^2+589024r+998761$, $1781332252r^2+1239131712r+2101097467$
116	$[-2,0,-1]$	34	$(-r+1)_2(-2r+1)_{17}(1)$	$4592r^2+3192r+5417$, $274400r^2+190876r+323659$
116	$[-2,0,-1]$	38	$(-r+1)_2(2r+1)_{19}(1)$	$82921r^2+57626r+97746$, $54599355r^2+37980374r+64400978$
116	$[-2,0,-1]$	38	$(2r+1)_{19}(-r+1)_2(1)$	$1081r^2+746r+1266$, $66555r^2+46310r+78482$
116	$[-2,0,-1]$	58	$(-r+1)_2(r^2+r-3)_{29}(1)$	$22024r^2+15320r+25977$, $-4956678r^2-3447968r-5846447$
135	$[-1,3,0]$	55	$(r^2+2)_{11}(r^2+1)_5(1)$	$4139r^2-19599r+18615$, $2077971r^2-1764501r+352796$
135	$[-1,3,0]$	88	$(r^2-r+2)_{11}(2)_8(1)$	$-1751r^2-1226r+577$, $-131901r^2-120528r+52524$
139	$[2,1,-1]$	46	$(r-3)_{23}(-r)_2(1)$	$22560r^2+19560r+1033$, $-8413992r^2+2336724r+7421723$
139	$[2,1,-1]$	57	$(r-1)_3(-2r+1)_{19}(1)$	$18r^2+61r+39$, $296r+239$
139	$[2,1,-1]$	57	$(-2r+1)_{19}(r-1)_3(1)$	$258r^2+541r+279$, $-17136r^2-9280r-3767$
140	$[-2,2,0]$	25	$(r^2+1)_5(r+1)_5(1)$	$1488r^2+992r+3968$, $110440r^2+88352r+287144$
140	$[-2,2,0]$	70	$(r^2+r+1)_7(r+1)_5(r)_2$	$139012r^2+106502r+360441$, $-1006136441r^2-77548384r-260995189$
140	$[-2,2,0]$	95	$(r^2+1)_5(r^2+2r+3)_{19}(1)$	$16r^2+16r$, $-64r^2+240r-120$
140	$[-2,2,0]$	95	$(r^2+1)_5(r^2+2r+3)_{19}(1)$	$64r^2-64r+48$, $-824r^2-368r+616$
172	$[3,-1,-1]$	45	$(r-2)_5(r^2-r-1)_9(1)$	$-1072r^2-80r+1872$, $-49976r^2-48864r+25920$
175	$[-3,2,-1]$	27	$(r)_3(r^2-r+2)_9(1)$	$-384r^2+816r-416$, $5904r^2-32472r+31816$
199	$[-1,4,-1]$	21	$(-r^2+r-2)_7(r^2-r+3)_3(1)$	$985297r^2+22348r-12672$, $-41881233r^2+130193546r-31313977$
199	$[-1,4,-1]$	21	$(-r^2+r-2)_7(r^2-r+3)_3(1)$	$-112647r^2-62978r+24321$, $-60304454r^2-96556295r+29529884$
199	$[-1,4,-1]$	33	$(r-2)_{11}(r^2-r+3)_3(1)$	$2802r^2+3055r-996$, $-398780r^2+635911r-139543$
199	$[-1,4,-1]$	49	$(-r^2+r-2)_7(-r^2-3)_7(1)$	$6447r^2-31223r+7758$, $3699375r^2-3171676r+577928$
199	$[-1,4,-1]$	49	$(-r^2-3)_7(-r^2+r-2)_7(1)$	$6447r^2-31223r+7758$, $3699375r^2-3171676r+577928$
199	$[-1,4,-1]$	77	$(r+1)_7(r-2)_{11}(1)$	$12952r^2-10791r+49899$, $2866751r^2-2163173r+10872899$
199	$[-1,4,-1]$	99	$(r-2)_{11}(r^2+1)_9(1)$	$-120r^2+576r-143$, $380r^2+4776r-1281$
200	$[2,2,-1]$	14	$(r+1)_2(r^2-r+1)_7(1)$	$-401r^2-3756r-2274$, $182521r^2-243668r-235802$
200	$[2,2,-1]$	14	$(r^2+r+1)_7(r+1)_2(1)$	$-241r^2+404r+366$, $5649r^2+3068r-394$
200	$[2,2,-1]$	65	$(-r^2-r-1)_{13}(-r^2+r-3)_5(1)$	$-1176r^2-1944r-767$, $75636r^2-142236r-124561$
204	$[-3,1,-1]$	21	$(r^2+r+1)_7(r)_3(1)$	$-48r^2+96r-32$, $-288r^2+1008r-872$
204	$[-3,1,-1]$	21	$(r^2+r+1)_7(r)_3(1)$	$262r^2-326r-44$, $-1784r^2-5128r+11612$
211	$[-3,-2,0]$	21	$(r+2)_7(-r)_3(1)$	$22010896r^2+41672992r+34877233$, $296072400488r^2+560550677168r+469139740087$
212	$[-2,4,-1]$	35	$(r^2-r+1)_7(r^2-r+3)_5(1)$	$29888r^2-13952r+112113$, $10054302r^2-4693580r+37714701$
216	$[-2,3,0]$	34	$(r)_2(r^2+r+5)_{17}(1)$	$307r^2+194r+1057$, $-11235r^2-6786r-37821$
216	$[-2,3,0]$	34	$(r^2+r+5)_{17}(r)_2(1)$	$307r^2+194r+1057$, $-11235r^2-6786r-37821$

$\lvert\Delta_K\rvert$	$f_K(x)$	$\mathrm{Nm}(\mathfrak{N})$	$\mathfrak{p}\mathfrak{D}\mathfrak{m}$	$c_4(E), c_6(E)$
216	$[-2,3,0]$	38	$(-2r^2-2r-7)_{19}(r)_2(1)$	$16r^2+81$, $-216r^2-192r-601$
231	$[3,0,-1]$	33	$(-r+1)_3(r^2-r+2)_{11}(1)$	$465r^2-1011r+1189$, $25273r^2-54957r+64546$
231	$[3,0,-1]$	33	$(r^2-r+2)_{11}(-r+1)_3(1)$	$465r^2-1011r+1189$, $25273r^2-54957r+64546$
231	$[3,0,-1]$	51	$(r^2+1)_{17}(r)_3(1)$	$-47r^2-50r+145$, $938r^2-291r+4$
239	$[-3,-1,0]$	24	$(r+1)_3(2)_8(1)$	$9r^2+18r+25$, $143r^2+236r+268$
239	$[-3,-1,0]$	57	$(-r^2-r+1)_{19}(r+1)_3(1)$	$1170r^2+1953r+2098$, $108233r^2+180929r+194227$
243	$[-3,0,0]$	10	$(r-2)_5(r-1)_2(1)$	$27576r^2+39771r+57360$, $11428272r^2+16482420r+23771763$
243	$[-3,0,0]$	10	$(r-1)_2(r-2)_5(1)$	$27576r^2+39771r+57360$, $11428272r^2+16482420r+23771763$
243	$[-3,0,0]$	22	$(r+2)_{11}(r-1)_2(1)$	$2002130917752r^2+2887572455827r+4164600133648$, $7076846143946804016r^2+10206578310238918020r+14720433182250993839$
243	$[-3,0,0]$	34	$(r^2+2)_{17}(r-1)_2(1)$	$10167352r^2+14663859r+21148944$, $-80986535280r^2-116802795708r-168458781921$
243	$[-3,0,0]$	46	$(-2r+1)_{23}(r-1)_2(1)$	$19946163r^2+28767345r+41489691$, $222892996797r^2+321467328855r+463636116909$
255	$[-3,0,-1]$	15	$(-r^2-1)_5(r-1)_3(1)$	$248r^2-320r-263$, $-2556r^2-4104r+16523$
255	$[-3,0,-1]$	15	$(-r^2-1)_5(r)_3(1)$	$19r^2-r+88$, $279r^2+908$
255	$[-3,0,-1]$	51	$(r^2-2)_{17}(r-1)_3(1)$	$-32r^2+240r-336$, $-3416r^2+2400r+7392$
255	$[-3,0,-1]$	51	$(r^2-2)_{17}(r-1)_3(1)$	$80r^2-80r-128$, $288r^2-2088r+2888$
255	$[-3,0,-1]$	65	$(r^2-r+1)_{13}(r+1)_5(1)$	$3r^2-105r+88$, $-909r^2+1116r-1576$
268	$[5,-3,-1]$	14	$(r^2-2)_7(r-1)_2(1)$	$-285113701784r^2-52062773310r+950706811227$, $144006413291532359r^2+50857254178772568r-433038348784793416$
300	$[-3,-3,-1]$	9	$(-r^2+2r+2)_3(r)_3(1)$	$26r^2+46r+4$, $504r^2+504r+460$
300	$[-3,-3,-1]$	9	$(-r^2+2r+2)_3(r)_3(1)$	$26r^2+46r+4$, $504r^2+504r+460$
300	$[-3,-3,-1]$	33	$(r^2-r-1)_{11}(r)_3(1)$	$11072r^2+17760r+12865$, $4675808r^2+7475664r+5398495$
300	$[-3,-3,-1]$	90	$(-r^2+2r+2)_3(-r-3)_{30}(1)$	-71, -1837
307	$[2,3,-1]$	10	$(r-1)_5(-r)_2(1)$	$-1450479r^2-118958r+338681$, $-1778021804r^2-7601175244r-3506038549$
307	$[2,3,-1]$	45	$(r-1)_5(r^2-2r+5)_9(1)$	$r^2+154r+81$, $-1744r^2-1756r-441$
324	$[-4,-3,0]$	4	$(r-2)_2(-r-1)_2(1)$	$345255874728r^2+758120909880r+628931968401$, $686899433218582980r^2+1508309811434747772r+1251283596457392135$
324	$[-4,-3,0]$	22	$(r^2-r-1)_{11}(r-2)_2(1)$	$808464801r^2+1775245884r+1472731953$, $77832295537635r^2+170905971571164r+141782435639127$
324	$[-4,-3,0]$	84	$(r^2+3r+3)_7(-r^2-3r-2)_{12}(1)$	$143742984r^2+315634200r+261847993$, $4700399015844r^2+10321245891900r+8562435635987$
327	$[-3,-2,-1]$	9	$(r)_3(r+1)_3(1)$	$13r^2+22r+25$, $144r^2+225r+242$
327	$[-3,-2,-1]$	15	$(-r+1)_5(r)_3(1)$	$1645r^2-2647r-2984$, $55543r^2-6268r-298328$
327	$[-3,-2,-1]$	15	$(r)_3(-r+1)_5(1)$	$1645r^2-2647r-2984$, $55543r^2-6268r-298328$
335	$[1,4,-1]$	25	$(r^2-r+3)_5(-r+1)_5(1)$	$-951r^2+1190r+57$, $61922r^2-78025r-346$
335	$[1,4,-1]$	25	$(r^2-r+3)_5(-r+1)_5(1)$	$10r^2-11r+10$, $52r^2-271r-29$
335	$[1,4,-1]$	65	$(-r^2+2r-4)_{13}(-r+1)_5(1)$	$61r^2-77r+247$, $101r^2-107r+380$
351	$[-3,3,0]$	33	$(r-2)_{11}(r)_3(1)$	$16r^2+144r-128$, $1824r^2-72r-1160$
356	$[7,1,-1]$	14	$(-r-2)_7(\frac{1}{2}r^2-r+\frac{3}{2})_2(1)$	$1577904r^2+58258032r+83210433$, $157810225239r^2+783843846012r+817040026548$
356	$[7,1,-1]$	26	$(-r+2)_{13}(-\frac{1}{2}r^2+r-\frac{5}{2})_2(1)$	$-353192r^2-495936r+44233$, $-560380445r^2-897785708r-94909392$
356	$[7,1,-1]$	26	$(-r+2)_{13}(-\frac{1}{2}r^2+r-\frac{5}{2})_2(1)$	$88412r^2+1393648r+1878333$, $112777386r^2+1758482408r+2367346473$
356	$[7,1,-1]$	196	$(r)_7(r-3)_{28}(1)$	$4182384r^2-3886864r-15048991$, $-37671142504r^2-30349104360r+38274580847$
364	$[-2,4,0]$	21	$(r-1)_3(-r-1)_7(1)$	$-368r^2-3712r+1840$, $-72736r^2+343360r-146264$
364	$[-2,4,0]$	26	$(-r^2-1)_{13}(-r)_2(1)$	$-266582r^2+148350r-10479$, $-274275343r^2+306719520r-83736937$
379	$[-4,1,-1]$	6	$(r-1)_3(-r+2)_2(1)$	$1418236432r^2+1053691808r+3254778265$, $137488390576232r^2+102148264969648r+315528648990403$
379	$[-4,1,-1]$	6	$(-r+2)_2(r-1)_3(1)$	$1418236432r^2+1053691808r+3254778265$, $137488390576232r^2+102148264969648r+315528648990403$
379	$[-4,1,-1]$	21	$(r-1)_3(r+1)_7(1)$	$15373338r^2+11421763r+35281005$, $-155147444344r^2-115268221468r-356055251669$
379	$[-4,1,-1]$	21	$(r+1)_7(r-1)_3(1)$	$15373338r^2+11421763r+35281005$, $-155147444344r^2-115268221468r-356055251669$
379	$[-4,1,-1]$	27	$(r-1)_3(r^2+1)_9(1)$	$1532208r^2+1138368r+3516337$, $1280550616r^2+951396864r+2938796535$
379	$[-4,1,-1]$	34	$(r-3)_{17}(-r+2)_2(1)$	$90342993r^2+67121158r+207332433$, $2363568298948r^2+1756034817652r+5424265343699$
439	$[5,-2,-1]$	15	$(-r+1)_3(r-2)_5(1)$	$-439r^2+1212r-1252$, $27743r^2-76494r+78935$
439	$[5,-2,-1]$	15	$(r-2)_5(-r+1)_3(1)$	$-439r^2+1212r-1252$, $27743r^2-76494r+78935$
440	$[-8,2,0]$	10	$(-r^2-2r-5)_5(-\frac{1}{2}r^2-r-2)_2(1)$	$-349392832r^2-1512227664r+3500497481$, $-128935660003280r^2-143880769408104r+276285496852283$
440	$[-8,2,0]$	10	$(-\frac{1}{2}r^2-r-2)_2(-r^2-2r-5)_5(1)$	$-349392832r^2-1512227664r+3500497481$, $-128935660003280r^2-143880769408104r+276285496852283$
440	$[-8,2,0]$	26	$(2r-3)_{13}(-\frac{1}{2}r^2-r-2)_2(1)$	$\frac{853}{2}r^2-6046r+8769$, $-\frac{419561}{2}r^2+835646r-810505$
451	$[8,-5,-1]$	26	$(2r-3)_{13}(-r+2)_2(1)$	$34296r^2+4776r-189951$, $5707476r^2+13155804r-1647297$
459	$[-8,3,0]$	22	$(\frac{1}{2}r^2-\frac{1}{2}r+1)_{11}(-\frac{1}{2}r^2-\frac{1}{2}r-2)_2(1)$	$16r^2-104r+121$, $-240r^2+1260r-1357$
459	$[-8,3,0]$	33	$(-\frac{1}{2}r^2-\frac{1}{2}r+1)_{11}(\frac{1}{2}r^2+\frac{1}{2}r+3)_3(1)$	$-\frac{19}{2}r^2+\frac{15}{2}r+21$, $-36r^2+96r-37$
459	$[-8,3,0]$	33	$(r^2+r+5)_{11}(\frac{1}{2}r^2+\frac{1}{2}r+3)_3(1)$	$\frac{178829}{2}r^2+\frac{270521}{2}r+472861$, $-83966694r^2-127020222r-444049333$
459	$[-8,3,0]$	34	$(\frac{1}{2}r^2+\frac{3}{2}r-3)_{17}(-\frac{1}{2}r^2-\frac{1}{2}r-2)_2(1)$	$\frac{125}{2}r^2+\frac{79}{2}r-59$, $282r^2-2430r+2691$
459	$[-8,3,0]$	44	$(\frac{1}{2}r^2-\frac{1}{2}r+1)_{11}(r-1)_4(1)$	$\frac{31}{2}r^2-\frac{103}{2}r+44$, $411r^2-1452r+1256$
459	$[-8,3,0]$	44	$(\frac{1}{2}r^2-\frac{1}{2}r+1)_{11}(r-1)_4(1)$	$\frac{103}{2}r^2-\frac{53}{2}r-60$, $237r^2+1374r-2984$
460	$[-3,5,-1]$	6	$(-r)_3(r-1)_2(1)$	$38808r^2+63978r-55637$, $28650959r^2+29220772r-29738968$
460	$[-3,5,-1]$	25	$(2r^2-r+10)_5(-r^2-4)_5(1)$	$36772r^2-83396r+37921$, $32322356r^2-98725758r+49331449$
460	$[-3,5,-1]$	26	$(r^2-r+1)_{13}(r-1)_2(1)$	$973808r^2-7106166r+4086627$, $-8777739333r^2+7426503436r-1197128148$
515	$[-4,-1,-1]$	14	$(-r+2)_2(r^2-2r-1)_7(1)$	$-7341361r^2-9117211r-13098483$, $-14436506787r^2-17928648161r-25757667905$
519	$[7,-4,-1]$	39	$(-r^2+3)_{13}(-r+2)_3(1)$	$-280r^2-960r-751$, $54220r^2+11272r-242353$
547	$[-4,-3,-1]$	14	$(-r+1)_7(r^2-2r-2)_2(1)$	$14509048r^2+24346088r+21671521$, $200457117220r^2+336365736396r+299413898447$
652	$[5,7,-1]$	14	$(-\frac{1}{2}r^2+r+\frac{1}{2})_7(-\frac{1}{2}r^2+r-\frac{9}{2})_2(1)$	$18r^2-36r+147$, $405r^2-648r+3294$
687	$[3,4,-1]$	9	$(r)_3(r+1)_3(1)$	$-7r^2+38r+25$, $-18r^2-423r-244$
743	$[-3,5,0]$	9	$(-r+1)_3(r)_3(1)$	$736r^2+416r+3913$, $-110256r^2-62192r-586373$
755	$[2,5,-1]$	10	$(-2r^2+3r-11)_5(-r)_2(1)$	$-1634r^2+10769r+4135$, $110372r^2+1174880r+412903$
815	$[-9,-7,0]$	9	$(r+1)_3(-r+3)_3(1)$	$26678105835217r^2+83793885354406r+76443429630973$, $717286463675094140331r^2+2252941797094015980448r+2055312234304678362824$
1196	$[-7,5,-1]$	14	$(-r)_7(-r+1)_2(1)$	$-12r^2-4r+25$, $-4r^2-134r+181$

Tables: quartic $(2, 1)$ fields (I)

$\lvert\Delta_K\rvert$	$f_K(x)$	$\mathrm{Nm}(\mathfrak{N})$	$\mathfrak{p}\mathfrak{D}\mathfrak{m}$	$c_4(E), c_6(E)$
643	$[1, -2, 0, -1]$	175	$(r^3 - r^2 - r - 1)_7(2r^3 - r^2 - 2)_{25}(1)$	$-1783r^3 + 1032r^2 + 522r + 3831$, $116369r^3 - 62909r^2 - 30125r - 248439$
688	$[-1, -2, 0, 0]$	11	$(-r^3 + r^2 + r + 2)_{11}(1)(1)$	$200r^3 + 284r^2 + 376r + 136$, $-5184r^3 - 7280r^2 - 10024r - 3672$
688	$[-1, -2, 0, 0]$	19	$(2r^3 - 3)_{19}(1)(1)$	$552r^3 + 764r^2 + 1064r + 392$, $-11536r^3 - 16160r^2 - 22584r - 8312$
731	$[-1, 0, 2, -1]$	80	$(r^2 + 1)_5(1)(2)_{16}$	$-848r^3 + 1529r^2 + 456r - 420$, $45471r^3 - 164824r^2 + 11648r + 72230$
775	$[-1, -3, 0, -1]$	176	$(-r^3 + r^2 + 1)_{11}(2)_{16}(1)$	$-\frac{6277}{2}r^3 - 2939r^2 - 5696r - \frac{3239}{2}$, $-528578r^3 - 495324r^2 - 959488r - 272875$
976	$[-1, 0, 3, -2]$	44	$(r - 2)_{11}(1)(r^3 - r^2 + r + 2)_4$	$-42r^3 - 21r^2 + 20r + 10$, $-10860r^3 - 12344r^2 + 6618r + 4899$
976	$[-1, 0, 3, -2]$	65	$(r^3 - 2r^2 + 4r)_{13}(1)(r^3 - 2r^2 + 3r + 1)_5$	$72r^3 + 20r^2 - 40r - 4$, $-1456r^3 + 3800r^2 - 176r - 1200$
1107	$[-1, -2, 0, -1]$	99	$(r - 1)_3(-2r + 1)_{33}(1)$	$105488r^3 + 90125r^2 + 152590r + 66821$, $120373437r^3 + 96189249r^2 + 171765105r + 67816591$
1156	$[1, -1, -2, -1]$	19	$(r^3 - r^2 - 2r - 3)_{19}(1)(1)$	$-816481030r^3 - 882631565r^2 - 203810962r + 392346684$, $-68032828897760r^3 - 73544780430596r^2 - 16982427384164r + 32692074898043$
1156	$[1, -1, -2, -1]$	19	$(r + 2)_{19}(1)(1)$	$-384131503r^3 - 415253582r^2 - 95887519r + 184588047$, $82379129020040r^3 + 89053403394404r^2 + 20563566138596r - 39585957243581$
1192	$[-1, 1, 2, -1]$	38	$(r^2 + 2)_{19}(1)(r^3 - r^2 + 2r)_2$	$9504r^3 + 11111r^2 - 4762r - 5690$, $-2387028r^3 + 7298060r^2 + 2454128r - 3005365$
1255	$[-1, -3, -1, 0]$	170	$(r^3 - r - 2)_2(-2r^3 + 2r^2 + 3)_{85}(1)$	$517916r^3 + 904037r^2 + 1060116r + 296716$, $-1433064139r^3 - 2501458160r^2 - 2933309166r - 820990264$
1423	$[-1, -2, 1, -1]$	98	$(r - 1)_2(2r^3 - r^2 + 2r - 2)_{49}(1)$	$39690531r^3 + 20246442r^2 + 70104884r + 26465314$, $702653466524r^3 + 356968363314r^2 + 1240909503739r + 466012978440$
1423	$[-1, -2, 1, -1]$	98	$(r^3 - r^2 + 2r - 1)_7(r^3 - 2r^2 + 2r - 1)_{14}(1)$	$54577r^3 + 27699r^2 + 96525r + 36260$, $1735232r^3 + 881975r^2 + 3066920r + 1151600$
1588	$[2, 0, -3, -1]$	56	$(-r^3 + r^2 + 3r + 1)_7(r^3 - r^2 - 3r)_8(1)$	$94560r^3 + 111816r^2 - 39672r - 86639$, $747493992r^3 + 883740564r^2 - 313920684r - 685060489$
1588	$[2, 0, -3, -1]$	152	$(r^3 - 3r - 1)_{19}(r^3 - r^2 - 3r)_8(1)$	$3496200r^3 + 4469800r^2 - 803168r - 2816543$, $26973722420r^3 + 32247663708r^2 - 10621228512r - 24308855297$
1600	$[-4, 0, -2, 0]$	11	$(\frac{1}{2}r^2 - r - 1)_{11}(1)(1)$	$12r^3 + 48r^2 + 56r + 20$, $-284r^3 - 460r^2 - 472r - 1024$
1600	$[-4, 0, -2, 0]$	11	$(\frac{1}{2}r^2 + r - 1)_{11}(1)(1)$	$276r^3 + 490r^2 + 336r + 628$, $-18172r^3 - 32652r^2 - 22424r - 40464$
1600	$[-4, 0, -2, 0]$	19	$(\frac{1}{2}r^3 - \frac{1}{2}r^2 - r - 1)_{19}(1)(1)$	$-44r^3 + 112r^2 - 56r + 148$, $-1660r^3 + 2572r^2 - 2056r + 3136$
1600	$[-4, 0, -2, 0]$	19	$(-\frac{1}{2}r^3 - \frac{1}{2}r^2 + r - 1)_{19}(1)(1)$	$44r^3 + 112r^2 + 56r + 148$, $1660r^3 + 2572r^2 + 2056r + 3136$
1732	$[-1, 3, 0, -1]$	13	$(r - 2)_{13}(1)(1)$	$3455801r^3 + 1359008r^2 - 3314187r + 836393$, $7590438778r^3 - 14215787438r^2 - 23508658710r + 9402560739$
1732	$[-1, 3, 0, -1]$	182	$(r^3 - r + 3)_7(r^2 - r - 2)_{26}(1)$	$-17184648r^3 - 14365296r^2 + 9302744r - 813151$, $93038140030r^3 - 219828160822r^2 - 331159079722r + 135298016971$
1823	$[-2, 3, 0, -1]$	114	$(-r^3 + r - 3)_3(r^3 + r^2 + 2)_{38}(1)$	$233810r^3 - 9696r^2 - 336273r + 159951$, $-70457084r^3 - 403468159r^2 - 171041003r + 342434077$
1879	$[1, -3, -2, -1]$	140	$(\frac{1}{2}r^3 - 2r - \frac{1}{2})_7(r^3 - r^2 - r - 2)_{20}(1)$	$-2436r^3 - 3240r^2 - 2688r + 1045$, $-49029102r^3 - 65262564r^2 - 54075240r + 21032621$

Tables: quartic $(2,1)$ fields (II)

$\lvert\Delta_K\rvert$	$f_K(x)$	Nm($\mathfrak{N}$)	p$\mathfrak{D}$m	$c_4(E), c_6(E)$
2051	$[1,3,-1,-1]$	15	$(r^3-r^2+2)_5(1)(-r+1)_3$	$-489r^3+1228r^2-1242r+18$, $46792r^3-100917r^2+73440r+47160$
2068	$[1,3,-2,-1]$	7	$(r-2)_7(1)(1)$	$269094497r^3+20141314r^2-35624307r-11296953$, $2473030585576r^3-3168333376r^2-6562955609992r-182979737393$
2068	$[1,3,-2,-1]$	13	$(-r^3+r^2+2r-1)_{13}(1)(1)$	$34500648r^3+3814392r^2-84122424r-23737447$, $-774084488074r^3-354426093238r^2-4154744686618r-92161502469$
2068	$[1,3,-2,-1]$	56	$(r-2)_7(r^3-2r+1)_8(1)$	$-3576591826r^3-1882130113r^2+6123537074r+1835712204$, $321001991693952r^3+322520099276304r^2-2812633044453488r-100176319060369$
2068	$[1,3,-2,-1]$	182	$(r-2)_7(-r^3+r^2-2)_{26}(1)$	$-1994707423r^3-282234694r^2+4755878517r+1346474783$, $-87331555991 62r^3-541369885659 86r^2-715946600083402r-16347374680241$
2092	$[-2,-3,1,-1]$	8	$(r)_2(1)(r-1)_4$	$-3r^3+29r^2-r+75$, $231r^3-61r^2+497r-287$
2096	$[2,-2,-2,0]$	28	$(r^3-r-1)_7(1)(r^3+r^2-2)_4$	$116r^3-390r^2+402r-94$, $7354r^3+222r^2-29620r+17640$
2116	$[-2,0,1,-1]$	5	$(r^2+1)_5(1)(1)$	$129712r^3+31248r^2+168480r+209073$, $-109612390r^3-26402860r^2-142375012r-176669575$
2116	$[-2,0,1,-1]$	130	$(r^2+1)_5(r+2)_{26}(1)$	$105064r^3+25312r^2+136464r+169353$, $78278092r^3+18855232r^2+101675032r+126166043$
2116	$[-2,0,1,-1]$	130	$(r^3-r^2+r+1)_{13}(r^3+r)_{10}(1)$	$105064r^3+25312r^2+136464r+169353$, $78278092r^3+18855232r^2+101675032r+126166043$
2183	$[-1,1,3,-2]$	126	$(-r^3+2r^2-4r)_7(r^3-2r^2+4r+1)_{18}(1)$	$-330539r^3-223654r^2+72664r+52816$, $421344240r^3+649688112r^2-51218957r-170790474$
2191	$[-1,0,3,-1]$	70	$(-r^3-2r-2)_5(-2r^3+r^2-5r-2)_{14}(1)$	$-928r^3+6929r^2-312r-2120$, $-885775r^3+1164640r^2+179150r-336602$
2191	$[-1,0,3,-1]$	80	$(-r^3-2r-2)_5(2)_{16}(1)$	$-408r^3+2689r^2-105r-821$, $120899r^3+70135r^2-44492r-24989$
2243	$[-1,-3,-1,-1]$	75	$(r-1)_5(-r^3+2r^2-r+2)_{15}(1)$	$5869003597r^3+6945285887r^2+929522310r+268803085$, $63399246832324r^3+75025661482408r^2+100410590521972r+29037147633615$
2243	$[-1,-3,-1,-1]$	75	$(r^3-r^2-2r-2)_5(-r^3+r^2+2r+3)_{15}(1)$	$5869003597r^3+6945285887r^2+929522310r+268803085$, $63399246832324r^3+75025661482408r^2+100410590521972r+29037147633615$
2243	$[-1,-3,-1,-1]$	105	$(-r^3+2r^2+2)_7(-r^3+2r^2-r+2)_{15}(1)$	$4336158r^3+5131353r^2+6867535r+1985981$, $-22914354769r^3+27116483373r^2+36291344215r-10494880213$
2243	$[-1,-3,-1,-1]$	105	$(r-1)_5(r^2-r+1)_{21}(1)$	$920025r^3+1088737r^2+1457115r+421377$, $3942374598r^3+4665343442r^2+6243862193r+1805625754$
2284	$[-4,2,2,-2]$	22	$(-r^2+r+1)_{11}(1)(\frac{1}{2}r^3-r^2+1)_2$	$-4322076r^3+3371584r^2-4531104r-14171719$, $-293858698818r^3+229234508344r^2-308070583688r-963537590781$
2327	$[-2,-1,-1,0]$	48	$(r^2-1)_3(2)_{16}(1)$	$60947675662300r^3+95467421346487r^2+88590894936957r+77819621400035$, $1595218950381053851625r^3+2498724287457442364789r^2+2318740987988175420378r+2036818157516553727423$
2327	$[-2,-1,-1,0]$	66	$(r^2-1)_3(-r^3+2)_{22}(1)$	$24654r^3+41044r^2+36631r+33971$, $13602419r^3+21481224r^2+19830770r+17549287$
2327	$[-2,-1,-1,0]$	78	$(r^2-1)_3(r^3-r^2+r)_{26}(1)$	$1632339r^3+2556895r^2+2372706r+2084241$, $5442997756r^3+8525820467r^2+7911705090r+6949764691$
2443	$[-1,-3,0,0]$	63	$(r^3-2)_3(-r-2)_{21}(1)$	$51601r^3+81980r^2+123695r+33695$, $-38870055r^3-59926714r^2-92404714r-25325630$
2443	$[-1,-3,0,0]$	117	$(-r^3+r^2-r+2)_{13}(-r^2+1)_9(1)$	$-26624r^3-78583r^2+147974r+56321$, $41156101r^3-906363r^2-80062921r-24803969$
2480	$[-2,-2,0,0]$	17	$(-r^3+r^2+r+1)_{17}(1)(1)$	$8r^3-12r^2-12r+17$, $212r^3+628r^2-818r-887$
2480	$[-2,-2,0,0]$	19	$(-r^2+r-1)_{19}(1)(1)$	$-648r^3+524r^2-408r+1636$, $29224r^3-23272r^2+18616r-73216$
2608	$[-2,-2,-2,0]$	50	$(-r+1)_5(r^3-r^2-r)_{10}(1)$	$-18122952r^3+23309952r^2+6270652r+28184369$, $-178706675384r^3+229835084602r^2+61821736238r+277904169213$
2696	$[1,-3,0,-1]$	24	$(r^3-2)_3(r^3-3)_8(1)$	$25999152r^3+20125515r^2+35704342r-14654974$, $-282591287516r^3-218749239468r^2-388079405968r+159288610195$
2816	$[-1,-4,-2,0]$	15	$(r^2-r-1)_5(1)(r^3-r^2-2r-1)_3$	$134184108r^3-165313100r^2-203588440r-41893502$, $2470282983044r^3-3964870336170r^2-2128766125800r-223343175430$
2859	$[-3,3,-1,-1]$	7	$(-r^3+r-1)_7(1)(1)$	$-4976r^3+12905r^2-15523r+9529$, $1469059r^3-3794717r^2+4539759r-2782843$
3119	$[-4,-3,-2,-1]$	23	$(\frac{2}{3}r^3-r^2-\frac{1}{3}r-\frac{1}{3})_{23}(1)(1)$	$16743632r^3+25416768r^2+30512064r+26598352$, $-406345115512r^3-616830291616r^2-740486023984r-645505557528$
3188	$[2,-4,1,-1]$	24	$(-r^3+r^2-r+3)_3(-r^3+r^2-r+4)_8(1)$	$2788172026368r^3+1423837175512r^2+4939120830288r-3691304019543$, $-10952993228320557238r^3-5593370421245480720r^2-19402732864546458324r+14500836945256233797$
3216	$[3,0,-1,-2]$	5	$(-r-1)_5(1)(1)$	$16r^3-40r^2+48r-20$, $104r^3-376r^2+816r-608$
3271	$[-1,-1,3,0]$	110	$(r^3+r^2+3r+2)_5(-r^3+r^2-2r+2)_{22}(1)$	$228r^3-115r^2+220r+132$, $6359r^3-2608r^2-6398r-1760$
3275	$[-9,6,2,-1]$	19	$(-\frac{1}{9}r^3-\frac{2}{9}r^2-\frac{8}{9}r-\frac{7}{3})_{19}(1)(1)$	$\frac{23}{3}r^3-\frac{20}{3}r^2+\frac{34}{3}r-14$, $\frac{496}{9}r^3-\frac{1690}{9}r^2+\frac{1979}{9}r-\frac{263}{3}$
3275	$[-9,6,2,-1]$	19	$(r-2)_{19}(1)(1)$	$3r^3+30r^2-27r-10$, $\frac{332}{9}r^3-\frac{632}{9}r^2-\frac{7136}{9}r+\frac{2693}{3}$
3284	$[-2,0,-1,-1]$	6	$(r-1)_3(1)(r)_2$	$2016r^3+1720r^2+1160r+2161$, $-290488r^3-248004r^2-169132r-313401$
3407	$[-3,1,-2,-1]$	84	$(\frac{1}{2}r^3-2r+\frac{1}{2})_7(r^2-r)_{12}(1)$	$\frac{28129013}{2}r^3+15057426r^2+3048856r+\frac{40754921}{2}$, $-125734882980r^3-134611455788r^2-27256382584r-182171881573$
3475	$[-11,8,-2,-1]$	11	$(-\frac{1}{7}r^3-\frac{2}{7}r^2-\frac{4}{7}r+\frac{1}{7})_{11}(1)(1)$	$\frac{61}{7}r^3-\frac{214}{7}r^2-\frac{351}{7}r+\frac{905}{7}$, $-\frac{1632}{7}r^3+\frac{2420}{7}r^2+\frac{6940}{7}r-\frac{10751}{7}$
3475	$[-11,8,-2,-1]$	11	$(-r)_{11}(1)(1)$	$-16r^3-8r^2+24r-95$, $\frac{9008}{7}r^3+\frac{5948}{7}r^2-\frac{8124}{7}r+\frac{59025}{7}$

Tables: quartic $(2,1)$ fields (III)

$\lvert\Delta_K\rvert$	$f_K(x)$	Nm($\mathfrak{N}$)	$\mathfrak{pDm}$	$c_4(E), c_6(E)$
3559	$[-2,-1,3,-2]$	20	$(r^2-1)_5(1)(-r^2+r-2)_4$	$266r^3-251r^2+481r+402$, $-8721r^3+6359r^2-17561r-13594$
3571	$[3,5,-5,-1]$	45	$(r^2-3)_3(r^2+r-5)_{15}(1)$	$-247r^3+1481r^2+5186r+1954$, $313052r^3+544864r^2-374892r-240173$
3632	$[2,-2,0,-2]$	13	$(r^3-r^2-r-1)_{13}(1)(1)$	$110352r^3+24580r^2+54624r-99300$, $124669648r^3+27763200r^2+61709112r-112178880$
3632	$[2,-2,0,-2]$	14	$(-r-1)_7(1)(-r)_2$	$2474r^3+522r^2+1234r-2217$, $523532r^3+116757r^2+258994r-471065$
3632	$[2,-2,0,-2]$	26	$(r^3-r^2-3)_{13}(1)(-r)_2$	$10028r^3+2232r^2+4964r-9023$, $-3562482r^3-793354r^2-1763358r+3205557$
3723	$[-1,3,1,-1]$	7	$(r^3-r^2+2r+2)_7(1)(1)$	$381r^3-208r^2-592r+201$, $-9752r^3-3598r^2+7307r-1656$
3723	$[-1,3,1,-1]$	17	$(r-2)_{17}(1)(1)$	$168r^3-1126r^2-1303r+504$, $-24313r^3+42209r^2+63347r-22837$
3775	$[-11,7,0,-1]$	19	$(\frac{3}{8}r^3-\frac{1}{4}r^2+\frac{1}{4}r+\frac{3}{8})_{19}(1)(1)$	$36r^3+8r^2+32r+253$, $-662r^3-228r^2-448r-5011$
3775	$[-11,7,0,-1]$	19	$(\frac{3}{8}r^3-\frac{1}{4}r^2+\frac{1}{4}r+\frac{27}{8})_{19}(1)(1)$	$-17r^3+30r^2-70r+8$, $\frac{489}{2}r^3-607r^2+1283r-\frac{1633}{2}$
3888	$[3,-6,0,-2]$	3	$(\frac{1}{2}r^3-\frac{1}{2}r^2-\frac{1}{2}r-\frac{5}{2})_3(1)(1)$	$12362r^3+8406r^2+22518r-13842$, $7035016r^3+4781484r^2+12812832r-7875996$
3899	$[-3,1,2,-2]$	23	$(r^3-2r^2+r+1)_{23}(1)(1)$	$-14r^3+14r^2-25r-21$, $381r^3-249r^2+364r+978$
3967	$[1,5,-2,-1]$	13	$(\frac{1}{2}r^3-2r+\frac{1}{2})_{13}(1)(1)$	$\frac{3321}{2}r^3+1456r^2-2668r-\frac{1081}{2}$, $-163448r^3+28056r^2+583644r+107371$
3967	$[1,5,-2,-1]$	17	$(\frac{1}{2}r^3-2r+\frac{5}{2})_{17}(1)(1)$	$-\frac{3537}{2}r^3-125r^2+4948r+\frac{1841}{2}$, $-99064r^3-306744r^2-273576r-41165$
4108	$[-2,-2,0,-1]$	52	$(r^2-r+1)_{13}(-r^3+2r^2-r+2)_4(1)$	$-52r^3+56r^2+316r+177$, $3676r^3-2050r^2-1438r+1283$
4192	$[-2,-2,1,0]$	28	$(r^2+r+1)_7(r^2+r+2)_4(1)$	$68388r^3-97900r^2-25440r+47889$, $50048814r^3-57380110r^2-27657416r+22526745$
4192	$[-2,-2,1,0]$	44	$(r^3+r-1)_{11}(r^2+r+2)_4(1)$	$993568r^3-1182928r^2-521264r+485673$, $-2157501576r^3+964037714r^2+2148667444r+353613881$
4204	$[-4,-2,0,0]$	20	$(-r+1)_5(-r)_4(1)$	$145360531282796r^3+161931312392192r^2-390193058066092r-440654493862007$, $-1159392135670300645002r^3+9949620873463783912066r^2-2497558503469317783050r-17611520739674724691341$
4319	$[2,-1,-4,-1]$	42	$(r)_2(-r^3+2r^2+3r-1)_{21}(1)$	$2626337501r^3+4156522706r^2+229413693r-2033846625$, $694511908654437r^3+1099155960247844r^2+60666438159866r-537832894958445$
4384	$[-4,0,3,-2]$	10	$(\frac{1}{2}r^3-r^2+\frac{5}{2}r-2)_5(1)(\frac{1}{2}r^3+\frac{1}{2}r+2)_2$	$-39342r^3+91445r^2+10340r-83032$, $-8399230r^3-58605841r^2+42062128r+73787052$
4423	$[1,4,-3,-1]$	50	$(-r+2)_5(-r^2-r+2)_{10}(1)$	$-4642767r^3-1724885r^2+13234188r+2913911$, $-19031399895r^3-11910891879r^2+44594523793r+10072542896$
4423	$[1,4,-3,-1]$	50	$(-r+2)_5(-r^2-r+2)_{10}(1)$	$3516856r^3+2151917r^2-8338704r-1880324$, $-9366159063r^3+477887546r^2+34591729866r+7408649776$
4564	$[1,-5,0,-1]$	5	$(\frac{1}{2}r^3+r-\frac{3}{2})_5(1)(1)$	$-280r^3+240r^2+64r+1449$, $10942r^3-8954r^2-1978r-55513$
4568	$[-1,-3,2,-1]$	12	$(r^2+2)_3(1)(r^2+3)_4$	$10845937505r^3+4588202505r^2+28221044093r+7621698413$, $-1760006389370257r^3-744542896235865r^2-4579522784006957r-1236798376628657$
4652	$[2,5,-3,-1]$	44	$(\frac{1}{2}r^3-\frac{3}{2}r+2)_{11}(-\frac{1}{2}r^3+\frac{3}{2}r-1)_4(1)$	$-1938032413r^3+62742964314r^2+143570326721r+41574563255$, $14844318169935843r^3+50626339684931473r^2+49275897864569564r+11502761970012547$
4663	$[2,-5,2,-1]$	11	$(-2r^3+r^2-3r+9)_{11}(1)(1)$	$4296r^3+1705r^2+10968r-6148$, $-3722961r^3-1477666r^2-9510026r+5330364$
4775	$[-9,-9,2,-1]$	11	$(-\frac{5}{12}r^3+\frac{2}{3}r^2-\frac{5}{6}r+\frac{13}{4})_{11}(1)(1)$	$\frac{307}{4}r^3-499r^2-\frac{2771}{2}r-\frac{2953}{4}$, $-\frac{69064}{3}r^3+\frac{146785}{3}r^2-\frac{182723}{3}r-87279$
4775	$[-9,-9,2,-1]$	11	$(\frac{1}{6}r^3+\frac{1}{3}r^2+\frac{1}{3}r-\frac{1}{2})_{11}(1)(1)$	$247r^3+539r^2-163r-335$, $-41241r^3-13659r^2-21597r-27719$
4832	$[-2,-4,-1,0]$	17	$(-r^3+r^2+2r+5)_{17}(1)(1)$	$-24r^3+17r^2+12r+82$, $580r^3-347r^2-524r-1770$
4907	$[-1,-4,-2,-1]$	11	$(r^3-r^2-3r-3)_{11}(1)(1)$	$-191405r^3-287504r^2-336559r-76491$, $1214356660r^3+1824081112r^2+2135314036r+485335595$
4944	$[-1,-4,-1,0]$	17	$(r^3-4)_{17}(1)(1)$	$316049736r^3+586633069r^2+772824316r+170272429$, $-23749113529508r^3-44081717952580r^2-58072797643568r-12794882314805$
4979	$[1,-3,-1,-1]$	13	$(-r^3+r^2+r+1)_{13}(1)(1)$	$32r^3-128r^2+144r-32$, $-1464r^3+3856r^2-1824r+240$

Surfaces (joint with X. Guitart and H. Sengun)

- This all generalizes to higer dimensional (e.g. 2-dim'l) components.
- The pairing
$$H^1(\Gamma_0(\mathfrak{N}),\mathbb{Z}) \times H_1(\Gamma_0(\mathfrak{N}),\mathbb{Z}) \to \mathbb{C}_p^\times$$
yields, by taking bases of the irreducible factors, a lattice $\Lambda \subset (\mathbb{C}_p^\times)^2$.
- Should correspond to the $\mathbb{C}_p$-points of an abelian surface A split at p.
- From the lattice Λ one can compute the p-adic L-invariant $\mathcal{L}_p$ of a Mumford–Schottky genus 2 curve.
  - $\mathcal{L}_p \in \mathbb{T} \otimes_{\mathbb{Z}} \mathbb{Q}_p$.
  - Corresponding to a hyperelliptic curve X with $\operatorname{Jac} X = A$.
- We can use formulas of Teitelbaum (1988) to recover a Weierstrass equation for X from $\mathcal{L}_p$.
- From this equation $\rightsquigarrow$ approximate Igusa invariants of X.
- Algebraic recognition algorithms $\rightsquigarrow$ actual Igusa invariants.

Toy example: an abelian surface over $F = \mathbb{Q}$

- Consider the Shimura curve $X(15 \cdot 11)$, which has genus $g = 9$.
- One of the factors J of $\operatorname{Jac} X(15 \cdot 11)$ is two-dimensional.
- T_2 acts on J with characteristic polynomial $P_2(x) = x^2 + 2x - 1$.
- We compute a basis $\{\phi_1, \phi_2\}$ of $H^1(\Gamma_0(15 \cdot 11), \mathbb{Z})^{T_2 = P_2}$ and a "pseudo-dual basis" $\{\theta_1, \theta_2\}$ of $H_1(\Gamma_0(15 \cdot 11), \mathbb{Z})^{T_2 = P_2}$.
- The integration pairing yields a symmetric matrix

$$\begin{pmatrix} \int_{\theta_1} \phi_1 & \int_{\theta_2} \phi_1 \\ \int_{\theta_1} \phi_2 & \int_{\theta_2} \phi_2 \end{pmatrix} = \begin{pmatrix} A & B \\ B & D \end{pmatrix}.$$

$$A = 3 \cdot 11^4 + 3 \cdot 11^5 + 4 \cdot 11^6 + \cdots + O(11^{24})$$
$$B = 4 + 7 \cdot 11 + 7 \cdot 11^2 + 4 \cdot 11^3 + \cdots + O(11^{20})$$
$$D = 9 \cdot 11^4 + 9 \cdot 11^5 + 8 \cdot 11^6 + 9 \cdot 11^7 + \cdots + O(11^{21})$$

Toy example: an abelian surface over $F = \mathbb{Q}$ (II)

- This allows to recover the 11-adic $\mathcal{L}$-invariant:

$$\mathcal{L}_{11}(J_2) = 3\cdot 11 + 8\cdot 11^2 + 3\cdot 11^4 + 9\cdot 11^5 + \cdots \\ + (7\cdot 11^2 + 3\cdot 11^3 + 5\cdot 11^4 + 2\cdot 11^5 + \cdots)\cdot T_2 \in \mathbb{T}\otimes\mathbb{Q}_p.$$

- We recover the Igusa–Clebsch invariants

$$(I_2 : I_4 : I_6 : I_{10}) = (2584 : -75356 : 37541976 : 2^{12}3^4 5^3 11^3)$$

- Mestre's algorithm (together with model reduction) yields the hyperelliptic curve

$$y^2 = -x^6 + 4x^4 - 10x^3 + 16x^2 - 9$$

- After twisting (by -1 in this case) we get a curve whose first few Euler factors match with those obtained by the $\mathbb{T}$-action on J.

Example surface over cubic $(1,1)$ field

- Let $F = \mathbb{Q}(r)$, where $r^3 - r^2 + 2r - 3 = 0$.
- Let $\mathfrak{p}_7 = (r^2 - r + 1)$.
- Let $B_{/F}$ be the totally definite quaternion algebra of disc. $\mathfrak{q}_3 = (r)$.
- $\operatorname{Jac} X^B(\Gamma_0(\mathfrak{p}))$ has a two-dimensional factor J.
- T_2 acts on J with characteristic polynomial $P_2(x) = x^2 + x - 10$.
- A similar calculation as before recovers the 7-adic $\mathcal{L}$-invariant:

$$\begin{aligned}\mathcal{L}_7(J) = 4 \cdot 7 + 2 \cdot 7^2 + 5 \cdot 7^3 + 3 \cdot 7^4 + 3 \cdot 7^5 + \cdots + O(7^{300}) \\ + (7^2 + 6 \cdot 7^4 + 2 \cdot 7^5 + 2 \cdot 7^6 + \cdots + O(7^{300})) \cdot T_2 \in \mathbb{T} \otimes \mathbb{Q}_7.\end{aligned}$$

- Sadly, haven't yet been able to recover Igusa–Clebsch invariants.
  - (We have about a dozen more examples over cubic and quartic fields. . .)

Beyond degree 1

- $F =$ quartic totally-complex field, $\mathfrak{N} = \mathfrak{p}$ (for simplicity).
- In this setting $\Gamma = \mathrm{SL}_2(\mathcal{O}_F[1/\mathfrak{p}])$, which acts on $\mathbb{H}_3^2$.
- The relevant groups are now $H_2(\Gamma, \mathrm{Div}^0 \mathcal{H}_\mathfrak{p})$ and $H^2(\Gamma, \Omega^1_{\mathcal{H}_\mathfrak{p}})$.
  - $H_2(\mathrm{SL}_2(\mathcal{O}_F), \mathrm{Div}\, \mathcal{H}_\mathfrak{p})$ & $H^2(\Gamma_0(\mathfrak{p}), \mathbb{D}) \cong H^2(\mathrm{SL}_2(\mathcal{O}_F), \mathrm{coInd}\, \mathbb{D})$.
  - The algorithms of J. Voight and A. Page do not extend to this situation.
- What did the modular symbols algorithm teach us?
  - Exploit the cusps…
  - …use **sharblies**!

Sharblies and overconvergent leezarbs

- Have a short exact sequence of $\mathrm{GL}_2(F)$-modules

$$0 \to \Delta^0 \to \mathrm{Div}\,\mathbb{P}^1(F) \stackrel{\deg}{\to} \mathbb{Z} \to 0, \quad \Delta^0 = \mathrm{Div}^0\,\mathbb{P}^1(F)$$

- Applying the functors $(-)\otimes V$ or $\mathrm{Hom}(-,W)$ and taking homology and cohomology yields connecting homomorphisms

$$\begin{array}{ccccc} H_2(\Gamma_0(\mathfrak{p}),V) & \times & H^2(\Gamma_0(\mathfrak{p}),W) & \xrightarrow{\cap} & H^0(\Gamma_0(\mathfrak{p}),V\otimes W) \\ \downarrow \delta & & \uparrow \delta & & \uparrow \mathrm{ev}_* \\ H_1(\Gamma_0(\mathfrak{p}),\Delta^0\otimes V) & \times & H^1(\Gamma_0(\mathfrak{p}),\mathrm{Hom}(\Delta^0,W)) & \xrightarrow{\cap} & H^0(\Gamma_0(\mathfrak{p}),X) \end{array}$$

$$X = \Delta^0\otimes V\otimes \mathrm{Hom}(\Delta^0,W) \stackrel{\mathrm{ev}}{\to} V\otimes W, \quad \gamma\otimes v\otimes\phi \mapsto v\otimes\phi(\gamma).$$

- This diagram is “compatible”:

$$\theta\cap\delta(\phi) = \mathrm{ev}_*(\delta\theta\cap\phi).$$

- Reduced to $H_1(\Gamma_0(\mathfrak{p}),\Delta^0\otimes V)$ and $H^1(\Gamma_0(\mathfrak{p}),\mathrm{Hom}(\Delta^0,W))$.

Sharblies and overconvergent leezarbs (II)

Reduced to $H_1(\Gamma_0(\mathfrak{p}), \Delta^0 \otimes V)$ and $H^1(\Gamma_0(\mathfrak{p}), \mathrm{Hom}(\Delta^0, W))$.

- Sharblies were invented by Szczarba and Lee (szczarb-lee) to compute with $H_1(\Gamma_0(\mathfrak{p}), \Delta^0 \otimes V)$.
  - Natural generalization of modular symbols to higher rank groups.
    - Ash–Rudolph, Ash–Gunnells, ...
  - Used to compute structure as hecke modules.
    - Ash, Gunnells, Hajir, Jones, McConnell, Yasaki, ...
  - They give an acyclic a resolution of $\Delta^0 \otimes V$.
  - The analogue of continued fractions algorithm is “sharbly reduction”.
- In order to compute $H^1(\Gamma_0(\mathfrak{p}), \mathrm{Hom}(\Delta^0, W))$, we introduce a dual version of sharblies, the **leezarbs**.
  - Leezarbs are an acyclic resolution of $\mathrm{Hom}(\Delta^0, W)$.
  - Can reuse the sharbly reduction algorithm to work with leezarbs.
  - This is work in progress with X. Guitart and A. Page, stay tuned!

Thank you !

and

Congratulations to the people of Catalunya

Who have realized that, in the course of human events, it has become necessary to dissolve the political bands which have connected them with Spain.