Quadratic twists of Siegel modular forms of paramodular level: Hecke operators and Fourier coefficients

Jennifer Johnson-Leung
University of Idaho

October 1, 2015

Joint work with Brooks Roberts

Recap

- Let L be a local field, (π, V) a representation of $\mathrm{GSp}(4, L)$, χ a quadratic character of $L^\times$.

Recap

- Let L be a local field, (π, V) a representation of $\mathrm{GSp}(4, L)$, χ a quadratic character of $L^\times$.
- In the last talk we constructed the twisting map $T_\chi(v) : V(0) \to V(4, \chi)$.

Recap

- Let L be a local field, (π, V) a representation of $\mathrm{GSp}(4, L)$, χ a quadratic character of $L^\times$.
- In the last talk we constructed the twisting map $T_\chi(v) : V(0) \to V(4, \chi)$.
- In this talk we consider the induced map on Siegel modular forms

$$\mathcal{T}_\chi : S_k(\Gamma^{\mathrm{para}}(N)) \to S_k(\Gamma^{\mathrm{para}}(Np^4)).$$

Siegel upper half plane

Let

$$J = \begin{pmatrix} & & 1 & \\ & & & 1 \\ -1 & & & \\ & -1 & & \end{pmatrix}.$$

Define the algebraic group GSp(4) to be the set of all $g \in \mathrm{GL}(4)$ such that ${}^t g J g = \lambda(g) J$ for some $\lambda \in \mathrm{GL}(1)$.
The group $\mathrm{GSp}(4, \mathbb{R})^+$ (where $\lambda(g) > 0$) acts on the Siegel upper half-plane

$$\mathfrak{H}_2 := \{Z = \begin{pmatrix} z & \tau \\ \tau & z' \end{pmatrix} \in \mathrm{Mat}(2 \times 2, \mathbb{C}) : \mathcal{I}(\mathrm{Z}) > 0\}$$

by transformations

$$g < Z >= (AZ + B)(CZ + D)^{-1}$$

where $g = \begin{pmatrix} A & B \\ C & D \end{pmatrix}$.

Paramodular Group

For N a positive integer we define

$$\Gamma^{\text{para}}(N) = \mathrm{Sp}(4, \mathbb{Q}) \cap \begin{bmatrix} \mathbb{Z} & \mathbb{Z} & N^{-1}\mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \\ N\mathbb{Z} & N\mathbb{Z} & \mathbb{Z} & N\mathbb{Z} \\ N\mathbb{Z} & \mathbb{Z} & \mathbb{Z} & \mathbb{Z} \end{bmatrix}.$$

Notice that $\Gamma^{\text{para}}(N) \not\subset \Gamma^{\text{para}}(Np)$.

Paramodular Forms

A *Paramodular form*, is a holomorpic function on $\mathfrak{H}_2$ such that for every

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{\text{para}}(N),$$

$$F|_k g(Z) = \lambda(g)^k \det(CZ+D)^{-k} F(g < Z >) = F(Z).$$

$S_k(\Gamma^{\text{para}}(N))$ be the space of Siegel modular cusp forms of weight k, degree two, and paramodular level N.

Fourier Coefficients

F has a Fourier expansion

$$F(Z) = \sum_{S \in A(N)^+} a(S) e^{2\pi i \mathrm{tr}(SZ)}$$

for $Z \in \mathfrak{H}_2$. Here, $A(N)^+$ is the set of all 2×2 matrices S of the form:

$$S = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}, \quad \alpha \in N\mathbb{Z}, \quad \gamma \in \mathbb{Z}, \quad \beta \in \frac{1}{2}\mathbb{Z}, \quad \alpha > 0,$$

$$\det \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix} = \alpha\gamma - \beta^2 > 0.$$

Let

$$g = \begin{bmatrix} A & \\ & {}^t A^{-1} \end{bmatrix}, \qquad A \in \Gamma_0(N).$$

Since $g \in \Gamma^{\mathrm{para}}(N)$, we have $F|_k(g) = F$ and so deduce that

$$a({}^t ASA) = \det(A)^k a(S).$$

Fourier Coefficients

- Since for
$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{\text{para}}(N),$$
$$F|_k g(Z) = \lambda(g)^k \det(CZ+D)^{-k} \sum_{S \in A(N)^+} a_\chi(S) e^{2\pi i \operatorname{tr}(S(AZ+B)(CZ+D)^{-1})}$$

Fourier Coefficients

- Since for

$$g = \begin{pmatrix} A & B \\ C & D \end{pmatrix} \in \Gamma^{\text{para}}(N),$$

$$F|_k g(Z) = \lambda(g)^k \det(CZ + D)^{-k} \sum_{S \in A(N)^+} a_\chi(S) e^{2\pi i \operatorname{tr}(S(AZ+B)(CZ+D)^{-1})}$$

- To have hope for a formula

$$\mathcal{T}_\chi(F)(Z) = \sum_{S \in A(Np^4)^+} a_\chi(S) e^{2\pi i \operatorname{tr}(SZ)}$$

with the $a_\chi(S)$ given in terms of the coefficients $a(S)$, we would like to have $C = 0$.

Iwasawa decomposition

- The Iwasawa decomposition asserts that $\mathrm{GSp}(4, L) = B \cdot \mathrm{GSp}(4, \mathfrak{o})$ where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GSp}(4, L)$ and $\mathfrak{o}$ is the ring of integers of L.

Iwasawa decomposition

- The Iwasawa decomposition asserts that $\mathrm{GSp}(4, L) = B \cdot \mathrm{GSp}(4, \mathfrak{o})$ where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GSp}(4, L)$ and $\mathfrak{o}$ is the ring of integers of L.
- If v is invariant under $\mathrm{GSp}(4, \mathfrak{o})$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices.

Iwasawa decomposition

- The Iwasawa decomposition asserts that $\mathrm{GSp}(4, L) = B \cdot \mathrm{GSp}(4, \mathfrak{o})$ where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GSp}(4, L)$ and $\mathfrak{o}$ is the ring of integers of L.
- If v is invariant under $\mathrm{GSp}(4, \mathfrak{o})$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices.
- This is also possible in principle for $v \in V(1)$.

Iwasawa decomposition

- The Iwasawa decomposition asserts that $\mathrm{GSp}(4, L) = B \cdot \mathrm{GSp}(4, \mathfrak{o})$ where B is the Borel subgroup of upper-triangular matrices in $\mathrm{GSp}(4, L)$ and $\mathfrak{o}$ is the ring of integers of L.
- If v is invariant under $\mathrm{GSp}(4, \mathfrak{o})$, then it is possible to obtain a formula for $T_\chi(v)$ involving only upper-triangular matrices.
- This is also possible in principle for $v \in V(1)$.
- For the remainder of the talk, we fix an odd prime $p \nmid N$ and we let χ be the nontrivial quadratic Dirichlet character modulo p.

Local Twisting Operator

$$T_\chi(v) = q^3 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} \chi(ab)\pi(\begin{bmatrix}1&&&\\&1&&\\&x&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&-a\varpi^{-1}&b\varpi^{-2}&z\varpi^{-4}\\&1&&b\varpi^{-2}\\&&1&a\varpi^{-1}\\&&&1\end{bmatrix})\tau v\,da\,db\,dx\,dz$$

$$+ q^3 \int_{\mathfrak{o}}\int_{\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} \chi(ab)\pi(\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&-a\varpi^{-1}&b\varpi^{-2}&z\varpi^{-4}\\&1&&b\varpi^{-2}\\&&1&a\varpi^{-1}\\&&&1\end{bmatrix})\tau v\,da\,db\,dy\,dz$$

$$+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{o}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} \chi(ab)\pi(t_4\begin{bmatrix}1&&&\\&1&&\\&x&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&-a\varpi^{-1}&b\varpi^{-2}&z\varpi^{-3}\\&1&&b\varpi^{-2}\\&&1&a\varpi^{-1}\\&&&1\end{bmatrix})\tau v\,da\,db\,dx\,dz$$

$$+ q^2 \int_{\mathfrak{o}}\int_{\mathfrak{p}}\int_{\mathfrak{o}^\times}\int_{\mathfrak{o}^\times} \chi(ab)\pi(t_4\begin{bmatrix}1&&&\\&&1&\\&-1&&\\&&&1\end{bmatrix}\begin{bmatrix}1&&&\\&1&&\\&y&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&-a\varpi^{-1}&b\varpi^{-2}&z\varpi^{-3}\\&1&&b\varpi^{-2}\\&&1&a\varpi^{-1}\\&&&1\end{bmatrix})\tau v\,da\,db\,dy\,dz.$$

Plan of attack

- Try to directly provide an Iwasawa identity $g = bk$ where $g \in \mathrm{GSp}(4, F)$, $b \in B$, and $k \in \mathrm{GSp}(4, \mathfrak{o})$.

Plan of attack

- Try to directly provide an Iwasawa identity $g = bk$ where $g \in \mathrm{GSp}(4, F)$, $b \in B$, and $k \in \mathrm{GSp}(4, \mathfrak{o})$.
- Use the formal matrix identity

$$\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & \\ & -x \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}.$$

Plan of attack

- Try to directly provide an Iwasawa identity $g = bk$ where $g \in \mathrm{GSp}(4, F)$, $b \in B$, and $k \in \mathrm{GSp}(4, \mathfrak{o})$.
- Use the formal matrix identity

$$\begin{bmatrix} 1 & \\ x & 1 \end{bmatrix} = \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix} \begin{bmatrix} -x^{-1} & \\ & -x \end{bmatrix} \begin{bmatrix} & 1 \\ -1 & \end{bmatrix} \begin{bmatrix} 1 & x^{-1} \\ & 1 \end{bmatrix}.$$

- It is necessary to decompose the domains of integration, and this leads to a proliferation of terms.

Example of a lemma

Lemma

$$q^2\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\chi(ab)\pi(t_4\begin{bmatrix}1&&&\\&1&&\\&x&1&\\&&&1\end{bmatrix}\begin{bmatrix}1&-a\varpi^{-1}&b\varpi^{-2}&z\varpi^{-3}\\&1&&b\varpi^{-2}\\&&1&a\varpi^{-1}\\&&&1\end{bmatrix})\tau v\,da\,db\,dx\,dz$$

$$=q^2\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\chi(b)\eta\pi(\begin{bmatrix}1&&&\\&1&&\\&&1&\\z\varpi^3&&&1\end{bmatrix}\begin{bmatrix}1&x\varpi^{-2}&a\varpi^{-2}&b\varpi^{-3}\\&1&&a\varpi^{-2}\\&&1&-x\varpi^{-2}\\&&&1\end{bmatrix})v\,da\,db\,dx\,dz$$

$$+\,q\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\chi(b)\eta\pi(\begin{bmatrix}1&&&\\&1&&\\&&1&\\z\varpi^4&&&1\end{bmatrix}\begin{bmatrix}1&x\varpi^{-2}&a\varpi^{-2}&b\varpi^{-3}\\&1&&a\varpi^{-2}\\&&1&-x\varpi^{-2}\\&&&1\end{bmatrix})v\,da\,db\,dx\,dz$$

$$+\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\chi(b)\eta\pi(\begin{bmatrix}1&&&\\&1&&\\&&1&\\z\varpi^5&&&1\end{bmatrix}\begin{bmatrix}1&x\varpi^{-2}&a\varpi^{-2}&b\varpi^{-3}\\&1&&a\varpi^{-2}\\&&1&-x\varpi^{-2}\\&&&1\end{bmatrix})v\,da\,db\,dx\,dz$$

$$+\,q^{-1}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}}\int\limits_{\mathfrak{o}^\times}\int\limits_{\mathfrak{o}^\times}\chi(b)\eta\pi(\begin{bmatrix}1&&&\\&1&&\\&&1&\\z\varpi^6&&&1\end{bmatrix}\begin{bmatrix}1&x\varpi^{-2}&a\varpi^{-2}&b\varpi^{-3}\\&1&&a\varpi^{-2}\\&&1&-x\varpi^{-2}\\&&&1\end{bmatrix})v\,da\,db\,dx\,dz$$

The slash formula

Theorem

Let N and k be positive integers, p a prime with $p \nmid N$, and χ a nontrivial quadratic Dirichlet character mod p. Then, the local twisting map

$$\mathcal{T}_\chi : S_k(\Gamma^{\text{para}}(N)) \to S_k(\Gamma^{\text{para}}(Np^4)).$$

If $F \in S_k(\Gamma^{\text{para}}(N))$, then this map is given by the formula

$$\mathcal{T}_\chi(F) = \sum_{i=1}^{14} F|_k \mathcal{T}_\chi^i,$$

where the T_i are given below.

Slash formula

$$\mathcal{T}_\chi^1 = p^{-11} \sum_{\substack{a,b,x \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \in \mathbb{Z}/p^4\mathbb{Z}}} \chi(ab) U(\begin{bmatrix} zp^{-4} & -bp^{-2} \\ -bp^{-2} & -x^{-1}p^{-1} \end{bmatrix}) A(\begin{bmatrix} 1 & (a+xb)p^{-1} \\ & 1 \end{bmatrix}),$$

$$\mathcal{T}_\chi^2 = p^{-11} \sum_{\substack{a,b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x,y \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x,y \not\equiv 1 (p)}} \chi(abxy) U(\begin{bmatrix} -ab(1-(1-y)^{-1}x)p^{-3} & -ap^{-2} \\ -ap^{-2} & -ab^{-1}(1-x)^{-1}p^{-1} \end{bmatrix}) A(\begin{bmatrix} p & bp^{-1} \\ & 1 \end{bmatrix}),$$

$$\mathcal{T}_\chi^3 = p^{-6} \sum_{\substack{a \in (\mathbb{Z}/p^2\mathbb{Z})^\times \\ b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ z \in (\mathbb{Z}/p\mathbb{Z})^\times \\ z \not\equiv 1 (p)}} \chi(b(1-z)) U(\begin{bmatrix} -bp^{-3} & ap^{-2} \\ ap^{-2} & -a^2b^{-1}zp^{-1} \end{bmatrix}) A(\begin{bmatrix} p & \\ & 1 \end{bmatrix}),$$

$$\mathcal{T}_\chi^4 = p^{-10} \sum_{\substack{a \in \mathbb{Z}/p^4\mathbb{Z} \\ b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x \in (\mathbb{Z}/p^4\mathbb{Z})^\times}} \chi(b) U(\begin{bmatrix} (ax-bp)p^{-4} & ap^{-2} \\ ap^{-2} & \end{bmatrix}) A(\begin{bmatrix} p & xp^{-2} \\ & 1 \end{bmatrix})$$

$$\mathcal{T}_\chi^5 = p^{-9} \sum_{\substack{a,b \in (\mathbb{Z}/p^3\mathbb{Z})^\times \\ x \in \mathbb{Z}/p^3\mathbb{Z}}} \chi(b) U(\begin{bmatrix} (ax-b)p^{-3} & ap^{-2} \\ ap^{-2} & \end{bmatrix} A(\begin{bmatrix} p & xp^{-1} \\ & 1 \end{bmatrix}),$$

Slash formula

$$\mathcal{T}_\chi^6 = p^{-6} \sum_{\substack{a,b\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ x\in(\mathbb{Z}/p\mathbb{Z})^\times}} \chi(bx)U(\begin{bmatrix} b(1+xp)p^{-2} & ap^{-2} \\ ap^{-2} & a^2b^{-1}p^{-2} \end{bmatrix})A(\begin{bmatrix} p^2 & \\ & 1 \end{bmatrix}),$$

$$\mathcal{T}_\chi^7 = p^{-7} \sum_{\substack{a\in(\mathbb{Z}/p^3\mathbb{Z})^\times \\ b\in(\mathbb{Z}/p\mathbb{Z})^\times \\ z\in\mathbb{Z}/p^4\mathbb{Z}}} \chi(ab)U(\begin{bmatrix} zp^{-4} & bp^{-1} \\ bp^{-1} & \end{bmatrix})A(\begin{bmatrix} 1 & -ap^{-1} \\ & p \end{bmatrix}),$$

$$\mathcal{T}_\chi^8 = p^{-9} \sum_{\substack{a,b,z\in(\mathbb{Z}/p^3\mathbb{Z})^\times \\ z\not\equiv 1(p)}} \chi(abz(1-z))U(\begin{bmatrix} ab(1-z)p^{-3} & ap^{-1} \\ ap^{-1} & \end{bmatrix} A(\begin{bmatrix} p & bp^{-1} \\ & p \end{bmatrix}),$$

$$\mathcal{T}_\chi^9 = p^{-6} \sum_{\substack{a\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ b,x\in(\mathbb{Z}/p\mathbb{Z})^\times}} \chi(b)U(\begin{bmatrix} bp^{-1} & \\ & xp^{-1} \end{bmatrix})A(\begin{bmatrix} p^2 & a \\ & p \end{bmatrix}),$$

$$\mathcal{T}_\chi^{10} = p^{-6} \sum_{\substack{a\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ b\in(\mathbb{Z}/p\mathbb{Z})^\times}} \chi(b)U(\begin{bmatrix} bp^{-1} & \\ & \end{bmatrix})A(\begin{bmatrix} p^2 & a \\ & p^2 \end{bmatrix}),$$

Slash formula

$$\mathcal{T}_\chi^{11} = p^{-10} \sum_{\substack{a\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ b\in(\mathbb{Z}/p^4\mathbb{Z})^\times \\ x\in\mathbb{Z}/p^3\mathbb{Z} \\ z\in\mathbb{Z}/p^4\mathbb{Z}}} \chi(ab)U(\begin{bmatrix} zp^{-4} & (ap+xb)p^{-3} \\ (ap+xb)p^{-3} & xp^{-2} \end{bmatrix})A(\begin{bmatrix} 1 & bp^{-2} \\ & p^{-1} \end{bmatrix}),$$

$$\mathcal{T}_\chi^{12} = p^{-12} \sum_{\substack{y\in\mathbb{Z}/p^4\mathbb{Z} \\ a\in(\mathbb{Z}/p^4\mathbb{Z})^\times \\ b,z\in(\mathbb{Z}/p^3\mathbb{Z})^\times \\ z\not\equiv 1(p)}} \chi(abz(1-z))U(\begin{bmatrix} a(y-b(1-z)p)p^{-4} & yp^{-3} \\ yp^{-3} & a^{-1}(y+bp)p^{-2} \end{bmatrix})A(\begin{bmatrix} p & ap^{-2} \\ & p^{-1} \end{bmatrix}),$$

$$\mathcal{T}_\chi^{13} = p^{-6} \sum_{\substack{a\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ b\in(\mathbb{Z}/p^3\mathbb{Z})^\times \\ x\in(\mathbb{Z}/p\mathbb{Z})^\times}} \chi(bx)U(\begin{bmatrix} b(1+x)p^{-1} & ap^{-2} \\ ap^{-2} & a^2b^{-1}p^{-3} \end{bmatrix})A(\begin{bmatrix} p^2 & \\ & p^{-1} \end{bmatrix}),$$

$$\mathcal{T}_\chi^{14} = p^{-6} \sum_{\substack{a\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ b\in(\mathbb{Z}/p\mathbb{Z})^\times \\ x\in\mathbb{Z}/p^4\mathbb{Z}}} \chi(b)U(\begin{bmatrix} bp^{-1} & ap^{-2} \\ ap^{-2} & xp^{-4} \end{bmatrix})A(\begin{bmatrix} p^2 & \\ & p^{-2} \end{bmatrix}).$$

- How do you know that this is correct?

- How do you know that this is correct?
- How do you know there is not something simpler?

Method of Calculating Fourier Coefficients

$$(F|_k\mathcal{T}_\chi^1)(Z) = p^{-11} \sum_{S\in A(N)^+} a(S) \sum_{a,b,x\in(\mathbb{Z}/p^3\mathbb{Z})^\times} \chi(ab)$$

$$\Big(\sum_{z\in\mathbb{Z}/p^4\mathbb{Z}} e^{2\pi i \mathrm{tr}(S\begin{bmatrix} zp^{-4} & -bp^{-2} \\ -bp^{-2} & -x^{-1}p^{-1}\end{bmatrix})} \Big) e^{2\pi i\mathrm{tr}(S[\begin{bmatrix}1 & (a+xb)p^{-1} \\ & 1\end{bmatrix}]Z)}.$$

Then the inner sum is calculated as

$$\sum_{z\in\mathbb{Z}/p^4\mathbb{Z}} e^{2\pi i(-\gamma p^{-1}x^{-1}-2b\beta p^{-2}+\alpha z p^{-4})} = \begin{cases} p^4 e^{-2\pi i(\gamma p x^{-1}+2b\beta)p^{-2}} & \text{if } p^4 \mid \alpha, \\ 0 & \text{otherwise.}\end{cases}$$

After simplification and rearrangement we obtain

$$(F|_k\mathcal{T}_\chi^1)(Z) = p^{-1} \sum_{\substack{S\in A(Np^4)^+ \\ p|2\beta}} \sum_{a,b\in(\mathbb{Z}/p\mathbb{Z})^\times} a(S[\begin{bmatrix}1 & -(a+b)p^{-1} \\ & 1\end{bmatrix}])$$

$$\chi(ab)W(\chi, -\gamma+2\beta p^{-1}a)e^{2\pi i\mathrm{tr}(SZ)}.$$

Fourier Coefficient Theorem

The twist of F has the Fourier expansion

$$\mathcal{T}_\chi(F)(Z) = \sum_{S \in A(Np^4)^+} W(\chi) a_\chi(S) e^{2\pi i \mathrm{tr}(SZ)}$$

where $W(\chi) = \sum_{b \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(b) e^{2\pi i b p^{-1}}$ and $a_\chi(S)$ for $S = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is given as follows:

Fourier Coefficient Theorem

The twist of F has the Fourier expansion

$$\mathcal{T}_\chi(F)(Z) = \sum_{S \in A(Np^4)^+} W(\chi) a_\chi(S) e^{2\pi i \mathrm{tr}(SZ)}$$

where $W(\chi) = \sum_{b \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(b) e^{2\pi i b p^{-1}}$ and $a_\chi(S)$ for $S = \begin{bmatrix} \alpha & \beta \\ \beta & \gamma \end{bmatrix}$ is given as follows:
If $p \nmid 2\beta$ and $p^4 \mid \alpha$, then

$$a_\chi(S) = p^{1-k} \chi(2\beta) \sum_{b \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi(b) a(S[\begin{bmatrix} 1 & -bp^{-1} \\ & p \end{bmatrix}]).$$

Fourier Coefficient Theorem

If $p \mid\mid 2\beta$ and $p^4 \mid\mid \alpha$, then $a_\chi(S)$ is

$$p^{-1} \sum_{a,b \in (\mathbb{Z}/p\mathbb{Z})^\times} \chi\big(ab(2\beta p^{-1}a - \gamma)\big) a(S[\begin{bmatrix} 1 & -(a+b)p^{-1} \\ & 1 \end{bmatrix}])$$

$$+ p^{-1} \sum_{\substack{a,z \in (\mathbb{Z}/p\mathbb{Z})^\times \\ z \not\equiv 1 (p)}} \chi\big(az(1-z)(az\alpha p^{-4} - 2\beta p^{-1})\big) a(S[\begin{bmatrix} p^{-1} & -ap^{-2} \\ & p \end{bmatrix}])$$

$$- p^{-1}\chi(\alpha p^{-4}) a(S[\begin{bmatrix} p^{-2} & \\ & p^2 \end{bmatrix}]) + p^{k-2}\chi(-\alpha p^{-4}) a(S[\begin{bmatrix} p^{-1} & -xp^{-3} \\ & 1 \end{bmatrix}])$$

$$+ p^{k-2}\chi(-\gamma) a(S[\begin{bmatrix} 1 & yp^{-2} \\ & p^{-1} \end{bmatrix}]),$$

where $x\alpha p^{-4} \equiv 2\beta p^{-1} (p^2)$ and $y 2\beta p^{-1} \equiv -\gamma (p^2)$.

Fourier Coefficient Theorem

If $p \mid\mid 2\beta$ and $p^5 \mid \alpha$, then $a_\chi(S)$ is

$$p^{-1} \sum_{a,b\in(\mathbb{Z}/p\mathbb{Z})^\times} \chi\big(ab(2\beta p^{-1}a-\gamma)\big)a(S[\begin{bmatrix}1 & -(a+b)p^{-1}\\ & 1\end{bmatrix}])$$
$$+\,p^{k-2}\chi(-\gamma)a(S[\begin{bmatrix}1 & yp^{-2}\\ & p^{-1}\end{bmatrix}])$$
$$-\,p^{-1}\chi(2\beta p^{-1}) \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} \chi(a)a(S[\begin{bmatrix}p^{-1} & -ap^{-2}\\ & p\end{bmatrix}]),$$

where $y \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ is such that $2\beta p^{-1}y \equiv -\gamma(p^2)$.

If $p^2 \mid 2\beta$ and $p^4 \parallel \alpha$, then

$$a_\chi(S) = a'_\chi(S) + a''_\chi(S) + a'''_\chi(S) \tag{1}$$

where

$$a'_\chi(S) = (1-p^{-1})\chi(\gamma)a(S) - p^{-1}\chi(\gamma) \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times} a(S[\begin{bmatrix}1 & -bp^{-1}\\ & 1\end{bmatrix}])$$

$$+ p^{k-3} \sum_{\substack{b,x,y\in(\mathbb{Z}/p\mathbb{Z})^\times\\ x,y\not\equiv 1(p)}} \chi\big(y(1-x)(\alpha p^{-4}(\frac{y-x}{1-y})b^2 + 2\beta p^{-2}b - \gamma x^{-1})\big)a(S[\begin{bmatrix}p^{-1} & -bp^{-2}\\ & 1\end{bmatrix}])$$

$$+ p^{k-3}\big(\chi(\gamma) + p\chi(-4\det(S)p^{-4}\gamma) - \chi(-\alpha p^{-4})W(\mathbf{1}, 2\beta p^{-2})\big)a(S[\begin{bmatrix}p^{-1} & \\ & 1\end{bmatrix}])$$

$$+ p^{k-3}\chi(-\alpha p^{-4}) \sum_{x\in\mathbb{Z}/p\mathbb{Z}} W(\mathbf{1}, 2\beta p^{-2} - x\alpha p^{-4})a(S[\begin{bmatrix}p^{-1} & -xp^{-2}\\ & 1\end{bmatrix}])$$

$$+ p^{2k-4} \sum_{\substack{b\in(\mathbb{Z}/p^2\mathbb{Z})^\times\\ p^2|(\alpha p^{-4}b^2 - 2\beta p^{-2}b + \gamma)}} \sum_{\substack{z\in(\mathbb{Z}/p\mathbb{Z})^\times\\ z\not\equiv 1(p)}} \chi\big(z(1-z)(\gamma - z\alpha p^{-4}b^2)\big)a(S[\begin{bmatrix}p^{-1} & -bp^{-3}\\ & p^{-1}\end{bmatrix}])$$

$$- p^{-1}\chi(\alpha p^{-4}) \sum_{a\in(\mathbb{Z}/p\mathbb{Z})^\times} a(S[\begin{bmatrix}p^{-1} & -ap^{-2}\\ & p\end{bmatrix}])$$

$$+ p^{k-3}\chi(\alpha p^{-4})(p\chi(-4\det(S)p^{-4}) - W(\mathbf{1},\gamma))a(S[\begin{bmatrix}p^{-2} & \\ & p\end{bmatrix}])$$

$$+ (1-p^{-1})\chi(\alpha p^{-4})a(S[\begin{bmatrix}p^{-2} & \\ & p^2\end{bmatrix}]),$$

$$a''_\chi(S) = \begin{cases} p^{2k-4}\chi(\alpha p^{-4})W(\mathbf{1}, 4\det(S)p^{-5})a(S[\begin{bmatrix} p^{-2} & \\ & 1 \end{bmatrix}]) & \text{if } p^5 \mid \det(S) \text{ and } \chi(\gamma\alpha p^{-4}) = 1 \\ 0 & \text{otherwise,} \end{cases}$$

and

$$a'''_\chi(S) = \begin{cases} p^{3k-5}\chi(\alpha p^{-4})W(\mathbf{1}, 4p^{-6}\det(S))a(S[\begin{bmatrix} p^{-2} & -ap^{-3} \\ & p^{-1} \end{bmatrix}]) & \text{if } p^6 \mid 4\det(S) \\ & \text{and } p^8 \nmid 4\det(S) \\ \chi(\alpha p^{-4})\Big(p^{3k-5}(p-1)a(S[\begin{bmatrix} p^{-2} & -ap^{-3} \\ & p^{-1} \end{bmatrix}]) & \text{if } p^8 \mid 4\det(S) \\ \quad +p^{4k-6}a(S[\begin{bmatrix} p^{-2} & -bp^{-4} \\ & p^{-2} \end{bmatrix}])\Big) & \\ 0 & \text{otherwise,} \end{cases}$$

where $a \in (\mathbb{Z}/p\mathbb{Z})^\times$ satisfies $2\alpha p^{-4}a \equiv 2\beta p^{-2}(p)$ and $b \in (\mathbb{Z}/p^2\mathbb{Z})^\times$ satisfies $2\alpha p^{-4}b \equiv 2\beta p^{-2}(p^2)$. (If a or b does not exist, then the corresponding term is 0.)

If $p^2 \mid 2\beta$ and $p^5 \mid \alpha$, then

$$
\begin{aligned}
a_\chi(S) = {} & (1-p^{-1})\chi(\gamma)a(S) - p^{-1}\chi(\gamma) \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times} a(S[\begin{bmatrix}1 & -bp^{-1}\\ & 1\end{bmatrix}]) \\
& - p^{k-3} \sum_{b\in(\mathbb{Z}/p\mathbb{Z})^\times} \sum_{\substack{x\in(\mathbb{Z}/p\mathbb{Z})^\times \\ x\not\equiv 1(p)}} \chi\big((1-x)(2\beta b p^{-2} - \gamma x^{-1})\big) a(S[\begin{bmatrix}p^{-1} & -bp^{-2}\\ & 1\end{bmatrix}]) \\
& + p^{k-3}\chi(\gamma)(1-p+p\chi(-4\det(S)p^{-4}))a(S[\begin{bmatrix}p^{-1} & \\ & 1\end{bmatrix}]) \\
& - p^{2k-4}\chi(-\gamma) \sum_{\substack{b\in(\mathbb{Z}/p^2\mathbb{Z})^\times \\ p^2|(\alpha p^{-4}b^2 - 2\beta p^{-2}b+\gamma)}} a(S[\begin{bmatrix}p^{-1} & -bp^{-3}\\ & p^{-1}\end{bmatrix}]). \qquad (2)
\end{aligned}
$$

Perhaps the first formula is enough

Theorem (Saha)

Let $F \in S_k(\mathrm{Sp}(4,\mathbb{Z}))$ *be such that* $a(S) = 0$ *for all but finitely many matrices* S *such that* $4\det(S)$ *is odd and squarefree. Then* $F = 0$.

Theorem (Marzec)

Let F *be a nonzero paramodular cuspidal newform of square-free level* N *and even weight* $k > 2$. *Then* F *has infinitely many nonzero fundamental Fourier coefficients.*

Vanishing on the Mass Space

Corollary

Let the notation be as in our Theorem and assume that k is even and $N = 1$. If F is in the Maass space as defined in then $\mathcal{T}_\chi(F) = 0$.

This is another rather hairy calculation, but the upshot is that we have non-trivial cancellations between the fourteen terms. Further, the terms that cancel with eachother vary with S.

LMFDB tells us that it is nonzero in general!

- $N = 1$
- $p = 3$
- $F = \Upsilon 20$ in LMFDB.

$$\begin{aligned} a_\chi(\begin{bmatrix} 81 & 22 \\ 22 & 6 \end{bmatrix}) &= 3^{-19}\chi(44)\left(a(\begin{bmatrix} 81 & 39 \\ 39 & 19 \end{bmatrix}) - a(\begin{bmatrix} 81 & 12 \\ 12 & 2 \end{bmatrix})\right) \\ &= -3^{-19}\left(a(\begin{bmatrix} 1 & 0 \\ 0 & 18 \end{bmatrix}) - a(\begin{bmatrix} 2 & 0 \\ 0 & 9 \end{bmatrix})\right) \\ &= -3^{-19}(2256995864880 + 4329978670800) \\ &= -\frac{2256995864880}{1162261467}. \end{aligned}$$

Compatibility with Hecke Operators

Theorem

For every prime $\ell \neq p$ we have the following commutation relations for the paramodular Hecke operators and Atkin-Lehner operator:

$$\begin{aligned} T(1,1,\ell,\ell)\mathcal{T}_\chi =& \chi(\ell)\mathcal{T}_\chi T(1,1,\ell,\ell), \\ T(1,\ell,\ell,\ell^2)\mathcal{T}_\chi =& \mathcal{T}_\chi T(1,\ell,\ell,\ell^2), \\ \mathcal{T}_\chi(F)|_k U_\ell =& \chi(\ell)^{\mathrm{val}_\ell(N)}\mathcal{T}_\chi(F|_k U_\ell), \end{aligned}$$

for $F \in S_k(\Gamma^{\mathrm{para}}(N))$.

What about this example?

- Recall that Chris Poor showed us the pair of examples of conductor 954 that seem to be quadratic twists.

What about this example?

- Recall that Chris Poor showed us the pair of examples of conductor 954 that seem to be quadratic twists.
- The problem is that $954 = 2 * 3^2 * 53$ so it is not covered by our theory.

Thank You!