Tables of Paramodular Forms

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including work in progess with:
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Modular Forms and Curves of Low Genus: Computational Aspects
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Computational Aspects of Modularity

1. Part *I*. The Paramodular Conjecture.
2. Part *II*. Using Fourier-Jacobi expansions to make rigorous tables of paramodular forms. (Joint work with J. Breeding and D. Yuen.)
3. Part *III*. Using Fourier-Jacobi expansions to make heuristic tables of paramodular forms. (Joint work with D. Yuen.)

4. Our paramodular website exists: math.lfc.edu/∼yuen/paramodular

All elliptic curves $E/\mathbb{Q}$ are modular

Theorem (Wiles; Wiles and Taylor; Breuil, Conrad, Diamond and Taylor)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of elliptic curves $E/\mathbb{Q}$ with conductor N*
2. *normalized Hecke eigenforms $f \in S_2(\Gamma_0(N))^{\mathrm{new}}$ with rational eigenvalues.*

In this correspondence we have $L(E, s, \mathrm{Hasse}) = L(f, s, \mathrm{Hecke})$.

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- Eichler (1954) proved the first examples $L(X_0(11), s, \mathrm{Hasse}) = L(\eta(\tau)^2\eta(11\tau)^2, s, \mathrm{Hecke})$.

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- Shimura proved 2 implies 1.
- Weil added $N = N$.

All abelian surfaces $A/\mathbb{Q}$ are paramodular

Paramodular Conjecture (Brumer and Kramer 2009)

Let $N \in \mathbb{N}$. There is a bijection between

1. *isogeny classes of abelian surfaces $A/\mathbb{Q}$ with conductor N and endomorphisms $\mathrm{End}_{\mathbb{Q}}(A) = \mathbb{Z}$,*
2. *lines of Hecke eigenforms $f \in S_2(K(N))^{\mathrm{new}}$ that have rational eigenvalues and are not Gritsenko lifts from $J_{2,N}^{\mathrm{cusp}}$.*

In this correspondence we have

$$L(A, s, \text{Hasse-Weil}) = L(f, s, \mathrm{spin}).$$

Remarks

- The paramodular group of level N,

$$K(N) = \begin{pmatrix} * & N* & * & * \\ * & * & * & */N \\ * & N* & * & * \\ N* & N* & N* & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Q}), \qquad * \in \mathbb{Z},$$

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- New form theory for paramodular groups:
  Ibukiyama 1984; Roberts and Schmidt 2004, (LNM 1918).
- Grit : $J_{k,N}^{\mathrm{cusp}} \to S_k(K(N))$, the Gritsenko lift from Jacobi cusp forms of index N to paramodular cusp forms of level N is an advanced version of the Maass lift.

All abelian surfaces $A/\mathbb{Q}$ are paramodular

The Paramodular Conjecture again after the remarks

Paramodular Conjecture

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A glimpse of higher modularity?

A sequence of discrete subgroups

The L-groups of symplectic groups are orthogonal groups.

Consider the following special, stable, integral orthogonal groups of spinor norm one.

- $\Gamma_0(N) \cong \mathrm{S\hat{O}}^+_{\mathbb{Z}} \begin{pmatrix} & & 1 \\ & -2N & \\ 1 & & \end{pmatrix}$

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- $K(N) \cong \mathrm{S}\hat{\mathrm{O}}^+_{\mathbb{Z}}\begin{pmatrix} & & & & 1 \\ & & & 1 & \\ & & -2N & & \\ & 1 & & & \\ 1 & & & & \end{pmatrix}$ (Gritsenko, Nikulin 1998)

A glimpse of higher modularity?

$$Q = Q_{2n+1}(N) = \begin{pmatrix} & & & & & 1 \\ & & & & 1 & \\ & & & \iddots & & \\ & & -2N & & & \\ & \iddots & & & & \\ 1 & & & & & \\ \end{pmatrix}$$

$Q_{2n+1}(N) = M'M$; $\quad \mathcal{L} = M\mathbb{Z}^{2n+1}$; $\quad U$ is hyperbolic plane

$\left(Q_{2n+1}(N), \mathbb{Z}^{2n+1}\right) \cong (\cdot, \mathcal{L}) \cong U^n \oplus (-2N)$

$\mathrm{S}\hat{\mathrm{O}}_{\mathbb{Z}}^{+}(Q) =$

$\{g \in \mathrm{SL}_{2n+1}(\mathbb{Z}) : g'Qg = Q, \forall x \in \mathcal{L}^*, gx - x \in \mathcal{L},\ \mathrm{sp.\,nm.}(g) = 1\}$

A glimpse of higher modularity?

References

1. Pei-Yu Tsai, *On Newforms for Split Special Odd Orthogonal Groups.* (Harvard thesis: 2013)
2. Benedict Gross, *On the Langlands correspondence for symplectic motives.* (2015)

Definition of Siegel Modular Form

- Siegel Upper Half Space: $\mathcal{H}_n = \{Z \in M^{\text{sym}}_{n \times n}(\mathbb{C}) : \operatorname{Im} Z > 0\}$.
- Symplectic group: $\sigma = \left(\begin{smallmatrix} A & B \\ C & D \end{smallmatrix}\right) \in \operatorname{Sp}_n(\mathbb{R})$ acts on $Z \in \mathcal{H}_n$ by $\sigma \cdot Z = (AZ + B)(CZ + D)^{-1}$.
- $\Gamma \subseteq \operatorname{Sp}_n(\mathbb{R})$ such that $\Gamma \cap \operatorname{Sp}_n(\mathbb{Z})$ has finite index in Γ and $\operatorname{Sp}_n(\mathbb{Z})$
- Slash action: For $f : \mathcal{H}_n \to \mathbb{C}$ and $\sigma \in \operatorname{Sp}_n(\mathbb{R})$, $(f|_k\sigma)(Z) = \det(CZ + D)^{-k} f(\sigma \cdot Z)$.
- Siegel Modular Forms: $M_k(\Gamma)$ is the $\mathbb{C}$-vector space of holomorphic $f : \mathcal{H}_n \to \mathbb{C}$ that are "bounded at the cusps" and that satisfy $f|_k\sigma = f$ for all $\sigma \in \Gamma$.

Definition of Siegel Modular Form

- Cusp Forms:
  $S_k(\Gamma) = \{f \in M_k(\Gamma)$ that "vanish at the cusps"$\}$
- Fourier Expansion: $f(Z) = \sum_{T>0} a(T; f) e(\mathrm{tr}(ZT))$
- For paramodular groups: $n = 2$; $\Gamma = K(N)$;
  symmetric $T \in \begin{pmatrix} \mathbb{Z} & \frac{1}{2}\mathbb{Z} \\ \frac{1}{2}\mathbb{Z} & N\mathbb{Z} \end{pmatrix}$

Plus and Minus Spaces

Paramodular Fricke involution

There is a paramodular involution μ_N that splits spaces of paramodular forms into plus and minus spaces.

$$S_k\left(K(N)\right) = S_k\left(K(N)\right)^+ \oplus S_k\left(K(N)\right)^-$$

$$S_k\left(K(N)\right)^\epsilon = \{f \in S_k\left(K(N)\right) : f|_k\mu_N = \epsilon f\}$$

$$\mu_N = \frac{1}{\sqrt{N}} \begin{pmatrix} 0 & N & & \\ -1 & 0 & & \\ & & 0 & 1 \\ & & -N & 0 \end{pmatrix} \in \mathrm{Sp}_2(\mathbb{R})^{\mathrm{pr}}$$

Fourier-Jacobi expansions

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- Fourier expansion of paramodular form $f \in M_k(K(N))$ in coordinates: $f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) =$ $\sum_{n,r,m \in \mathbb{Z},\, n,m \geq 0,\, 4Nnm \geq r^2} a\left(\begin{pmatrix} n & r/2 \\ r/2 & Nm \end{pmatrix}; f\right) e(n\tau + rz + Nm\omega)$

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- Fourier-Jacobi expansion of paramodular form $f \in M_k(K(N))$: $f\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right) = \sum_{m\in\mathbb{Z},\, m\geq 0} \phi_{Nm}(\tau, z)e(Nm\omega)$
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  The FJE of paramodular form can be a more suggestive analogy to the elliptic case than the full Fourier expansion.
- Coefficients $\phi_{Nm} \in J_{k,Nm}$ are *Jacobi forms*.
  $\phi_{Nm}(\tau, z) = \sum_{n,r\in\mathbb{Z}:\, n\geq 0,\, 4Nnm\geq r^2} a\left(\begin{pmatrix} n & r/2\\ r/2 & Nm\end{pmatrix}; f\right)e(n\tau + rz)$

Fourier-Jacobi expansion (FJE)

$$\text{FJE: } f\left(\begin{smallmatrix} \tau & z \\ z & \omega \end{smallmatrix}\right) = \sum_{m\in\mathbb{Z},\, m\geq 0} \phi_{Nm}(\tau, z) e(Nm\omega)$$

The Fourier-Jacobi expansion of a paramodular form is fixed *term by term* by the following subgroup of the paramodular group $K(N)$:

$$P_{2,1}(\mathbb{Z}) = \begin{pmatrix} * & 0 & * & * \\ * & * & * & * \\ * & 0 & * & * \\ 0 & 0 & 0 & * \end{pmatrix} \cap \mathrm{Sp}_2(\mathbb{Z}), \qquad * \in \mathbb{Z},$$

- $P_{2,1}(\mathbb{Z})/\{\pm I\} \cong \mathrm{SL}_2(\mathbb{Z}) \ltimes \mathrm{Heisenberg}(\mathbb{Z})$

Thus the coefficients ϕ_{Nm} are automorphic forms in their own right and easier to compute than Siegel modular forms. This is one motivation for the introduction of Jacobi forms.

Fourier-Jacobi expansion
Two Natural Questions

$$\text{FJE: } f\begin{pmatrix} \tau & z \\ z & \omega \end{pmatrix} = \sum_{m\in\mathbb{Z},\, m\geq 0} \phi_{Nm}(\tau, z)e(Nm\omega)$$

We ask two natural questions:

1. Which Jacobi forms can be a Fourier-Jacobi coefficient of a paramodular form?
2. Can we find the consistency conditions among a sequence of Jacobi forms that characterize the Fourier-Jacobi expansions of paramodular forms?

Fourier-Jacobi expansion

Natural Question Number One

$$\text{FJE: } f\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right) = \sum_{m\in\mathbb{Z},\, m\geq 0} \phi_{Nm}(\tau, z)e(Nm\omega)$$

The existence of the Gritsenko lift answers the first question. Any Jacobi cusp form can be the *first* Fourier-Jacobi coefficient of a paramodular form.

Theorem (Gritsenko)

For $\phi \in J_{k,m}^{\mathrm{cusp}}$ the series $\mathrm{Grit}(\phi)$ converges and defines a map

$$\mathrm{Grit} : J_{k,m}^{\mathrm{cusp}} \to S_k\left(K(m)\right)^{\epsilon}, \quad \epsilon = (-1)^k.$$

$$\mathrm{Grit}(\phi)\left(\begin{smallmatrix}\tau & z\\ z & \omega\end{smallmatrix}\right) = \sum_{\ell\in\mathbb{N}} \ell^{2-k}(\phi|V_\ell)(\tau, z)e(\ell m\omega).$$

Fourier-Jacobi expansions

Natural Question Number Two

Fix $N \in \mathbb{N}$. Consider a formal series of Jacobi forms ϕ_{Nm} of weight k and index Nm:

$$\sum_{m \in \mathbb{N}} \phi_{Nm} \xi^{Nm}.$$

If this sequence of Jacobi forms is the Fourier-Jacobi expansion of a paramodular form f in $S_k(K(N))^{\epsilon}$, then the Fourier coefficients of the Jacobi forms satisfy simple relations that follow from the μ_N-eigenvalue condition on f:

$$c(n, r; \phi_{Nm}) = \epsilon(-1)^k c(m, r; \phi_{Nn})$$

Experimentally, these conditions seem to be sufficient as well, but this has only been proven for $N \leq 4$.

Natural Question Two
References

The sufficiency of the conditions $c(n, r; \phi_{Nm}) = \epsilon(-1)^k c(m, r; \phi_{Nn})$ is not hard to believe because $K(N)^+ = \langle K(N), \mu_N \rangle = \langle P_{2,1}(\mathbb{Z}), \mu_N \rangle$. The only obstruction is convergence.

1. Aoki, *Estimating Siegel modular forms of genus 2 using Jacobi forms* (2000) $(N = 1)$
2. Ibukiyama, Poor, Yuen, *Jacobi forms that characterize paramodular forms.* (2013) $(N \leq 4)$
3. Brunier, Raum, *Kudla's Modularity Conjecture and Formal Fourier-Jacobi Series* (2015)

Notation: $K(N)^* = \langle K(N), \text{All paramodular AL-involutions} \rangle$

Strategy for the rigorous computation of paramodular forms

1. Find a sufficient number of Fourier-Jacobi coefficients, say L, that determine a paramodular form in $S_k(K(N))^\epsilon$, *without* knowing the dimension of $S_k(K(N))^\epsilon$.
2. Use the theory of *theta blocks* due to Gritsenko, Skoruppa, and Zagier, to span spaces of Jacobi forms.
3. Use the conditions $c(n, r; \phi_{Nm}) = \epsilon(-1)^k c(m, r; \phi_{Nn})$ to define a vector space V containing all possible initial Fourier-Jacobi expansions of length L from $S_k(K(N))^\epsilon$.
4. If you can construct $\dim V$ linearly independent paramodular forms in $S_k(K(N))^\epsilon$ then you have proven $\dim S_k(K(N))^\epsilon = \dim V$.

How many Fourier-Jacobi coefficients determine a paramodular cusp form?

Definition

The Jacobsthal function $j(N)$ is defined to be the smallest positive integer m such that every sequence of m consecutive positive integers contains an integer coprime to N.

Examples: $j(2) = j(3) = j(4) = j(5) = 2$,
$j(6) = 4$, $j(10) = 4$, $j(15) = 3$.

$j(N) \in O((\ln N)^2)$; H. Iwaniec, *On the problem of Jacobsthal* (1978)

How many FJCs determine a paramodular form?

Theorem (Breeding, Poor, Yuen)

Let $k, N \in \mathbb{N}$. Let $\chi : K(N)^* \to \{\pm 1\}$ be a character trivial on $K(N)$. Let $f \in S_k(K(N)^*, \chi)$ be a common eigenfunction of the paramodular Atkin-Lehner involutions and have Fourier-Jacobi expansion

$$f = \sum_{j=1}^{\infty} \phi_{jN} \xi^{jN}.$$

Let $N = p_1^{\alpha_1} \cdots p_\ell^{\alpha_\ell}$ be the prime factorization of N and set $\tilde{N} = p_1 \cdots p_\ell$. Choose $\mu \in \mathbb{N}$ such that $2\mu + 1 \geq j(\tilde{N}/p_i)$ for all i. Let κ be 1 when N is prime, 2 when N is a composite prime power and $1 + \mu + \mu^2$ otherwise. If $\phi_{jN} = 0$ for $j \leq \kappa \frac{k}{10} N \prod_{p^r || N} \frac{p^r + p^{r-2}}{p^r + 1}$, then $f = 0$.

Rigorous dimensions of weight two paramodular forms

Table 1. Dimension of $S_2(K(N))$ and number of FJ-coefficients needed in proof for $N \leq 60$. Omitted levels N indicate that $\dim S_2(K(N)) = 0$.

N	37	43	53	57	58
dim	1	1	1	1^{++}	1^{++}
FJCs	7	8	10	9	8

These eigenforms are all Gritsenko lifts.
Paramodular Conjecture (vacuously) true for (odd) levels $N \leq 60$.

Rigorous dimensions of weight three paramodular forms

Table 2. Dimension of $S_3(K(N))$ and number of FJ-coefficients needed in proof for $N \leq 40$. Omitted levels N indicate that $\dim S_3(K(N)) = 0$.

N	13	17	19	21	22	23	25	26	27	28
dim	1	1	1	1^{+-}	1^{+-}	1	1	1^{+-}	1	1^{-+}
FJCs	3	4	5	4	3	6	8	5	7	4

N	29	31	32	33	34	35	37	38	39	40
dim	2	2	1	2^{+-}_{-+}	2^{+-2}	1^{+-}	4	2^{+-2}	2^{+-2}	1^{-+}
FJC	7	9	10	6	7	3	10	6	9	7

Strategy for heuristic computation of paramodular forms

Run everything in $\mathbb{F}_q$, for an auxillary prime q, to save memory.

1. Use enough Fourier-Jacobi coefficients to achieve insight but not rigor for $S_k(K(N))^\epsilon$.
2. Use the theory of *theta blocks* due to Gritsenko, Skoruppa, and Zagier, to span spaces of Jacobi forms.
3. Use the conditions $c(n, r; \phi_{Nm}) = \epsilon(-1)^k c(m, r; \phi_{Nn})$ to define a vector space V containing all possible initial Fourier-Jacobi expansions from $S_k(K(N))^\epsilon$ for a fixed **short** length.
4. *Hope* that $\dim S_k(K(N))^\epsilon = \dim V$.

Heuristic tables: $k = 2$ paramodular newforms: $N \leq 600$.

$$+\text{new} = \dim\left((S_2(K(N))^{\text{new}})^+ \,/\, \text{Grit}\left(J_{2,N}^{\text{cusp}}\right)\right)$$

$$-\text{new} = \dim\left(S_2(K(N))^{\text{new}}\right)^- .$$

N	$+$new	$-$new	various comments
249	≥ 1		ss-Jac
277	$= 1$		ss-Jac
295	≥ 1		ss-Jac
349	$= 1$		ss-Jac
353	$= 1$		ss-Jac
388	≥ 1		ss-Jac

N	$+$new	$-$new	various comments
389	$= 1$		ss-Jac
394	1		ss-Jac
427	1		ss-Jac
461	≤ 1		ss-Jac
464	1		ss-Jac
472	1		ss-Jac
511	2		unknown 4 dim $A/\mathbb{Q}$, quad pair, $\sqrt{5}$
523	≤ 1		ss-Jac
550	1		surface unknown

N	$+$new	$-$new	various comments
555	$=1$		ss-Jac
561	1		ss-Prym
574	1		ss-Jac
587	≤ 1	$=1$	both ss-Jac
597	1		ss-Jac
$\dots$			
657	1		Weil Res., $E/\mathbb{Q}(\sqrt{-3})$
775	1		Weil Res., $E/\mathbb{Q}(\sqrt{5})$
954	2		twists by $\sqrt{-3}$

Thanks to Armand Brumer for all the abelian surfaces in this table.

(And thank you to Andrew Sutherland for correcting some of my errors.)

Thank you!