From $K3$ Surfaces to Noncongruence Modular Forms

Explicit Methods for Modularity of $K3$ Surfaces and
Other Higher Weight Motives

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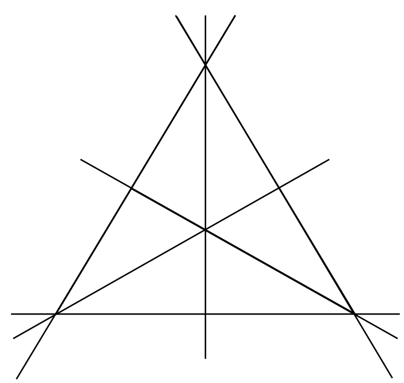
A $K3$ surface with rank 20

- There are thirteen $K3$ surfaces defined over $\mathbb{Q}$ whose NS group has rank 20, generated by algebraic cycles over $\mathbb{Q}$.
- Elkies-Schütt constructed them from suitable double covers of $\mathbb{P}^2$ branched above 6 lines.
- Consider such a $K3$ surface $\mathcal{E}_2$ constructed by Beukers and Stienstra the same way, with the 6 lines positioned as



- The zeta function at a good prime p has the form

$$Z(\mathcal{E}_2/\mathbb{F}_p, T) = \frac{1}{(1-T)(1-p^2T)P_2(T)},$$

where $P_2(T) =$ char. poly. of Frob_p on $H^2_{et}(\mathcal{E}_2 \otimes_{\mathbb{Q}} \overline{\mathbb{Q}}, \mathbb{Q}_\ell)$ is in $\mathbb{Z}[T]$ of degree 22.

- Beukers and Stienstra computed

$$P_2(T) = (1-pT)^{20} P(\mathcal{E}_2; p; T)$$

with $P(\mathcal{E}_2; p; T) \in \mathbb{Z}[T]$ of degree 2.

- They further showed that

$$L(\mathcal{E}_2, s) := \prod_p \frac{1}{P(\mathcal{E}_2; p; p^{-s})} = L(\eta(4z)^6, s)$$

is modular.

Elliptic surfaces

- $\mathcal{E}_2$ has a nonhomogeneous model in the sense of Shioda

$$\mathcal{E}_2: \quad y^2 + (1 - t^2)xy - t^2 y = x^3 - t^2 x^2$$

  with parameter t.

- For $n \geq 2$ consider the elliptic surface in the sense of Shioda

$$\mathcal{E}_n: \quad y^2 + (1 - t_n^{\,n})xy - t_n^{\,n} y = x^3 - t_n^{\,n} x^2$$

  parametrized by t_n. It is an n-fold cover of $\mathbb{P}^2$ branched above the same configuration of 6 lines.

- The Hodge diamond of $\mathcal{E}_n$ is of the form

$$\begin{array}{ccccc} & & 1 & & \\ & 0 & & 0 & \\ (n-1) & & 10n & & (n-1) \\ & 0 & & 0 & \\ & & 1 & & \end{array}$$

- The zeta of $\mathcal{E}_n/\mathbb{F}_p$ looks similar, with $\deg P_2(T) = 12n - 2$. $P_2(T)$ is a product of $10n$ linear factors, from points on algebraic cycles, and $P(\mathcal{E}_n; p; T) \in \mathbb{Z}[T]$ of degree $2n - 2$.
- Similarly define $L(\mathcal{E}_n, s) = \prod_p \frac{1}{P(\mathcal{E}_n; p; p^{-s})}$.

  Question: Is $L(\mathcal{E}_n, s)$ automorphic, i.e., equal to the L-function of an automorphic form?

Base curves as modular curves

- Beukers and Stienstra: The elliptic surface

$$\mathcal{E}: \quad y^2 + (1-\tau)xy - \tau y = x^3 - \tau x^2$$

parameterized by τ is fibered over the genus 0 modular curve (defined over $\mathbb{Q}$) of

$$\Gamma^1(5) = \left\{ \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in SL(2,\mathbb{Z}), \ \equiv \begin{pmatrix} 1 & 0 \\ * & 1 \end{pmatrix} \mod 5 \right\}.$$

- $\mathcal{E}_n$ is fibered over a genus zero n-fold cover X_n (defined over $\mathbb{Q}$) of $X_{\Gamma^1(5)}$ under $\tau = t_n^n$.

- $X_{\Gamma^1(5)}$ has no elliptic points, and 4 cusps $\infty, 0, -2, -5/2$. The matrix $A = \begin{pmatrix} -2 & -5 \\ 1 & 2 \end{pmatrix} \in \Gamma^0(5)$ normalizes $\Gamma^1(5)$, $A^2 = -Id$.

- Let E_1 be an Eisenstein series of weight 3 having simple zeros at all cusps except ∞, and $E_2 = E_1|A$. Then $\tau = \frac{E_1}{E_2}$ is a Hauptmodul for $\Gamma^1(5)$ with a simple zero at the cusp -2 and a simple pole at the cusp ∞.
  $A(\tau) = -1/\tau$ is an involution on $X_{\Gamma^1(5)}$.
- With $t_n = \sqrt[n]{\tau}$, the curve X_n is unramified over $X_{\Gamma^1(5)}$ except totally ramified above the cusps ∞ and -2 (with τ-coordinates ∞ and 0, resp.). This describes the index-n normal subgroup Γ_n of $\Gamma^1(5)$ such that X_n is the modular curve of Γ_n.
- $\mathcal{E}_n$ is the universal elliptic curve over X_n.
- Γ_n is noncongruence if $n \neq 5$.
- $S_3(\Gamma_n) =< (E_1^j E_2^{n-j})^{1/n} >_{1 \le j \le n-1}$ is $(n-1)$-dimensional, corresponding to holomorphic 2-differentials on $\mathcal{E}_n$.

Galois representations

- To $S_3(\Gamma_n)$, Scholl has attached a compatible $2(n-1)$-dimensional ℓ-adic representations $\rho_{n,\ell}$ of $G_{\mathbb{Q}} = Gal(\bar{\mathbb{Q}}/\mathbb{Q})$ acting on $W_{n,\ell} = H^1(X_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \iota_* \mathcal{F}_\ell)$, similar to Deligne's construction for congruence forms.
- He showed that $W_{n,\ell}$ can be embedded into $H^2_{et}(\mathcal{E}_n \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}, \mathbb{Q}_\ell)$ and the L-function attached to the family $\{\rho_{n,\ell}\}$ is $L(\mathcal{E}_n, s)$.
- According to Langlands philosophy, the family $\{\rho_{n,\ell}\}$ is conjectured to correspond to an automorphic representation of some reductive group. If so, call $\{\rho_{n,\ell}\}$ *automorphic*, and then $L(\mathcal{E}_n, s)$ is an automorphic L-function. Call $\{\rho_{n,\ell}\}$ *potentially automorphic* if there is a finite index subgroup G_K of $G_{\mathbb{Q}}$ such that $\{\rho_{n,\ell}|_{G_K}\}$ is automorphic.

Properties of Scholl representations $\rho_{n,\ell}$

1. $\rho_{n,\ell}$ is unramified outside $n\ell$;
2. For ℓ large, $\rho_{n,\ell}|_{G_{\mathbb{Q}_\ell}}$ is crystalline with Hodge-Tate weights 0 and -2, each with multiplicity $n-1$;
3. $\rho_{n,\ell}$(complex conjugation) has eigenvalues ± 1, each with multiplicity $n-1$;
4. The actions $A(t_n) = \frac{\zeta_{2n}}{t_n}$ and $\zeta(t_n) = \zeta_n^{-1} t_n$ on X_n, where $\zeta = \begin{pmatrix} 1 & 5 \\ 0 & 1 \end{pmatrix}$, induce actions on the space of $\rho_{n,\ell}$.

Since Serre's modularity conjecture is proved by Kahre-Wintenberger and Kisin in 2007, all degree 2 Scholl representations are modular. So $L(\mathcal{E}_2, s)$ is modular, as proved by Beukers-Stienstra.

Automorphy of $L(\mathcal{E}_3, s)$

This was proved by L-Long-Yang in 2005.

We computed the char. poly. of $\rho_{3,\ell}(\mathrm{Frob}_p)$ for small primes p and found them agree with those of $\tilde{\rho}_\ell := \rho_{g_+,\ell} \oplus \rho_{g_-,\ell}$, where $\rho_{g_\pm,\ell}$ are the ℓ-adic Deligne representations attached to the wt 3 newforms $g_\pm$ of level 27 quad. char. χ_{-3}:

$$\begin{aligned} g_\pm(z) = \; & q \mp 3iq^2 - 5q^4 \pm 3iq^5 + 5q^7 \pm 3iq^8 + \\ & +9q^{10} \pm 15iq^{11} - 10q^{13} \mp 15iq^{14} - \cdots \end{aligned}$$

To show them isomorphic, choose $\ell = 2$. The actions of A on $\rho_{3,2}$ and the Atkin-Lehner involution on $\tilde{\rho}_2$ allow both representations to be viewed as 2-dimensional representations over $\mathbb{Q}(i)_{1+i}$. Then Faltings-Serre was applied to prove $\rho_{3,2} \simeq \tilde{\rho}_2$, only used char. polys. at primes $5 \leq p \leq 19$.

Automorphy of $L(\mathcal{E}_4, s)$

This was proved by Atkin-L-Long in 2008 with conceptual explanation given in Atkin-L-Long-Liu in 2013.

The repn $\rho_{4,\ell} = \rho_{2,\ell} \oplus \rho^{-}_{4,\ell}$ as eigenspaces with eigenvalues ± 1 of ζ^2, where $\rho^{-}_{4,\ell}$ is 4-dim'l and want to prove it automorphic.

Its space admits quaternion multiplication by $B_{-2} := A(1+\zeta)$ and $B_2 := A(1-\zeta)$ defined over $\mathbb{Q}(\sqrt{\mp 2})$ resp., satisfying

$$(B_{-2})^2 = -2I = (B_2)^2 \quad \text{and} \quad B_{-2}B_2 = -B_2B_{-2}.$$

For each quadratic extension K in the biquadratic extension $F := \mathbb{Q}(\sqrt{2}, \sqrt{-1})$,

$$\rho^{-}_{4,\ell}|_{G_K} = \sigma_{K,\ell} \oplus (\sigma_{K,\ell} \otimes \delta_{F/K}),$$

where $\delta_{F/K}$ is the quadratic char. of F/K.

There is a finite character χ_K of G_K so that $\sigma_{K,\ell} \otimes \chi_K$ extends to a degree-2 representation $\eta_{K,\ell}$ of $G_{\mathbb{Q}}$ and

$$\rho^-_{4,\ell} = \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \sigma_{K,\ell} = \eta_{K,\ell} \otimes \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi_K^{-1}.$$

Both $\eta_{K,\ell}$ and $\mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi_K^{-1}$ are automorphic, and so is $\sigma_{K,\ell}$.

Now $L(\mathcal{E}_4, s) = L(\mathcal{E}_2, s) L(\rho^-_{4,\ell}, s)$, and there are 5 ways to see the automorphicity of $L(\rho^-_{4,\ell}, s)$:

$$L(\rho^-_{4,\ell}, s) = L(\sigma_{K,\ell}, s) \quad (GL(2) \text{ over three } K \subset \mathbb{Q}(\sqrt{2}, \sqrt{-1}))$$
$$= L(\eta_{K,\ell} \otimes \mathrm{Ind}_{G_K}^{G_{\mathbb{Q}}} \chi_K^{-1}, s) \quad (GL(2) \times GL(2) \text{ and } GL(4) \ \text{ over } \mathbb{Q}).$$

Similar argument applies to $L(\mathcal{E}_6, s)$, done by Long.

Computing $1/P(\mathcal{E}_n;p;T)$

Let $p \nmid n$. To compute $1/P(\mathcal{E}_n;p;T)$, we use a model birational to $\mathcal{E}_n$ over $\mathbb{Q}$ defined by the nonhomogeneous equation

$$s^n = (xy)^{n-1}(1-y)(1-x)(1-xy)^{n-1} =: f_n(x,y).$$

The points with $s = 0$ lie on algebraic cycles.

Let q be a power of p. The number of solutions to $s^n = f_n(x,y)$ over $\mathbb{F}_q$ with $s \neq 0$ is given by

$$\sum_{i=1}^{r} \sum_{x,y\in\mathbb{F}_q,\ f_n(x,y)\neq 0} \xi_r^i(f_n(x,y)),$$

where $r = gcd(n, q-1)$ and ξ_r is a character of $\mathbb{F}_q^\times$ of order r. The sums with $i \neq r$ contribute to $1/P(\mathcal{E}_n;p;T)$ and the sum with $i = r$ contributes to other factors of $Z(\mathcal{E}_n/\mathbb{F}_p,T)$.

Character sums and Galois representations

At a place $\wp$ of $\mathbb{Q}(\zeta_n)$ with residue field $k_\wp$ of cardinality q, n divides $q-1$. The nth power residue symbol at $\wp$, denoted $\left(\frac{}{\wp}\right)_n$, is a $<\zeta_n> \cup\{0\}$-valued function defined by

$$\left(\frac{a}{\wp}\right)_n \equiv a^{(q-1)/n} \pmod{\wp} \quad \text{for all } a \in \mathbb{Z}_{\mathbb{Q}(\zeta_n)}.$$

It induces a character of $k_\wp^\times$ with order n.

Fuselier-Long-Ramakrishna-Swisher-Tu show that, for $1 \le i \le n-1$ there exists a degree-2 representation $\sigma_{n,i,\ell}$ of $G_{\mathbb{Q}(\zeta_n)}$ such that at each place $\wp$ of $\mathbb{Q}(\zeta_n)$ where $\sigma_{n,i,\ell}$ is unramified, one has

$$\mathrm{Tr}\sigma_{n,i,\ell}(\mathrm{Frob}_\wp) = \sum_{x,y\in k_\wp} \left(\frac{f_n(x,y)}{\wp}\right)_n^i.$$

This gives the decomposition

$$\rho_{n,\ell}|G_{\mathbb{Q}(\zeta_n)} = \sigma_{n,1,\ell} \oplus \sigma_{n,2,\ell} \oplus \cdots \oplus \sigma_{n,n-1,\ell}.$$

ζ preserves each $\sigma_{n,i,\ell}$, while A sends $\sigma_{n,i,\ell}$ to $\sigma_{n,n-i,\ell}$.

Further, the character sum can be expressed as a finite field analogue of hypergeometric series, which was shown by Greene to equal to its complex conjugation up to sign, i.e.,

$$\mathrm{Tr}\sigma_{n,i,\ell}(\mathrm{Frob}_\wp) = \left(\frac{-1}{\wp}\right)_n^i \mathrm{Tr}\sigma_{n,n-i,\ell}(\mathrm{Frob}_\wp).$$

Therefore, either $\sigma_{n,i,\ell} \simeq \sigma_{n,n-i,\ell}$, or they differ by a quadratic twist.

Automorphy of $L(\mathcal{E}_n, s)$ revisited

(I) $n = 2$. $\mathbb{Q}(\zeta_2) = \mathbb{Q}$.

In this case $\sigma_{2,1,\ell} = \rho_{2,\ell}$ is the only representation.

The character is the Legendre symbol, which is the quadratic character χ_{-1} of $\mathbb{Q}(\sqrt{-1})$ over $\mathbb{Q}$.

This shows that $\rho_{2,\ell}$ is invariant under the quadratic twist by χ_{-1}, hence it is induced from a character of $G_{\mathbb{Q}(\sqrt{-1})}$.

It is modular and the corresponding weight 3 cusp form $\eta(4z)^6$ has CM, as observed by Beukers-Stienstra.

(II) $n = 3$. $\mathbb{Q}(\zeta_3) = \mathbb{Q}(\sqrt{-3})$.

There are two representations $\sigma_{3,1,\ell}$ and $\sigma_{3,2,\ell}$.

Since n is odd, at a place $\wp$ of $\mathbb{Q}(\sqrt{-3})$ not above 2, $q = \#k_\wp$ is odd so that $(q-1)/3$ is even and hence the sign is always 1. Thus $\sigma_{3,1,\ell} \simeq \sigma_{3,2,\ell}$. On the other hand, $\sigma_{3,2,\ell}$ is the conjugate of $\sigma_{3,1,\ell}$ by the nontrivial element in $Gal(\mathbb{Q}(\sqrt{-3})/\mathbb{Q})$, this means that $\sigma_{3,1,\ell}$ extends to a degree 2 representation of $G_\mathbb{Q}$, denoted by ρ_ℓ^+. Similarly $\sigma_{3,2,\ell}$ extends to a representation ρ_ℓ^- of $G_\mathbb{Q}$ so that

$$\rho_{3,\ell} = \rho_\ell^+ \oplus \rho_\ell^- .$$

Since $\rho_\ell^{\pm}$ have the same restrictions to $G_{\mathbb{Q}(\sqrt{-3})}$, they either agree of differ by the quadratic twist χ_{-3}. To determine which one, one computes $\mathrm{Tr}\rho_{3,\ell}(\mathrm{Frob}_p)$ at primes $p \equiv 2 \pmod 3$ by counting solutions to $s^3 = f_3(x,y) \pmod p$ with $s \neq 0$. Since $p \equiv 2 \pmod 3$, we have $r = gcd(3, p-1) = 1$. Thus $\mathrm{Tr}\rho_{3,\ell}(\mathrm{Frob}_p) = 0$ and $\rho_\ell^- = \rho_\ell^+ \otimes \chi_{-3}$. This explains why $g_\pm$ differ by twist by χ_{-3}.

(III) $n = 4$. $\mathbb{Q}(\zeta_4) = \mathbb{Q}(\sqrt{-1})$.

There are 3 representations: $\sigma_{4,2,\ell} = \rho_{2,\ell}$ studied before, and $\sigma_{4,1,\ell}$ and $\sigma_{4,3,\ell}$ summing to $\rho^-_{4,\ell}|G_{\mathbb{Q}(\sqrt{-1})}$.

Since n is even, the character $\left(\frac{-1}{\wp}\right)_4$ of $G_{\mathbb{Q}(\sqrt{-1})}$ has order 2 and kernel $G_{\mathbb{Q}(\sqrt{-1},\sqrt{2})}$. In other words, it is the quadratic character of $\mathbb{Q}(\sqrt{-1},\sqrt{2})$ over $\mathbb{Q}(\sqrt{-1})$. So $\sigma_{4,1,\ell}$ and its conjugate $\sigma_{4,3,\ell}$ are not isomorphic, and

$$\rho^-_{4,\ell} = \mathrm{Ind}_{G_{\mathbb{Q}(\sqrt{-1})}}^{G_{\mathbb{Q}}} \sigma_{4,1,\ell}$$

as discussed before.

Similar discussion applies to $n = 6$ case.

Potential automorphy of $L(\mathcal{E}_n, s)$

For each proper divisor d of n, $\rho_{n,\ell}$ naturally contains $\rho_{d,\ell}$ as a $G_{\mathbb{Q}}$-invariant direct summand. After removing the "old" part from $d|n$ and $d < n$, the remaining "new" part is denoted $\rho_{n,\ell}^{prim}$, which has dimension $2\phi(n)$. Thus

$$\rho_{n,\ell} = \sum_{d|n,\ d\neq 1} \rho_{d,\ell}^{prim}.$$

$\{\rho_{n,\ell}^{prim}\}$ remains a compatible family.

Assume $n \geq 7$. Then $\phi(n)$ is even. Denote by $\mathbb{Q}(\zeta_n)^+$ the totally real subfield of $\mathbb{Q}(\zeta_n)$. We have the decomposition

$$\rho_{n,\ell}^{prim}|_{G_{\mathbb{Q}(\zeta_n)}} = \sum_{1\leq i\leq n-1,\ (i,n)=1} \sigma_{n,i,\ell}.$$

Recall that $\sigma_{n,n-i,\ell}$ is the conjugate of $\sigma_{n,i,\ell}$ under the nontrivial element in $Gal(\mathbb{Q}(\zeta_n)/\mathbb{Q}(\zeta_n)^+)$.

First assume n odd. We have $\sigma_{n,i,\ell} \simeq \sigma_{n,n-i,\ell}$ and they both extend to degree-2 representations $\eta_{n,i,\ell}$ and $\eta_{n,n-i,\ell}$ of $G_{\mathbb{Q}(\zeta_n)^+}$ so that

$$\rho_{n,\ell}^{prim}|_{G_{\mathbb{Q}(\zeta_n)^+}} = \sum_{1\leq i\leq n-1,\ (n,i)=1} \eta_{n,i,\ell}.$$

Next assume n even. In this case $\sigma_{n,n-i,\ell}$ and $\sigma_{n,i,\ell}$ differ by a quadratic twist. Then

$$\sigma_{n,i,\ell} \oplus \sigma_{n,n-i,\ell} = \eta_{n,i,\ell} \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}(\zeta_n)^+}} \chi_{n,i,\ell}$$

for a degree-2 representation $\eta_{n,i,\ell}$ of $G_{\mathbb{Q}(\zeta_n)^+}$ and a finite character $\chi_{n,i,\ell}$ of $G_{\mathbb{Q}(\zeta_n)}$. Hence

$$\rho_{n,\ell}^{prim}|G_{\mathbb{Q}(\zeta_n)^+} = \sum_{1\leq i\leq \phi(n)/2,\ (n,i)=1} \eta_{n,i,\ell} \otimes \mathrm{Ind}_{G_{\mathbb{Q}(\zeta_n)}}^{G_{\mathbb{Q}(\zeta_n)^+}} \chi_{n,i,\ell}.$$

In both cases, $\{\eta_{n,i,\ell}\}$ is a compatible family for each i. In an on-going work L-Liu-Long, it is shown that $\eta_{n,i,\ell}$ is potentially automorphic, hence so is $\rho_{n,\ell}^{prim}$.

Theorem [L-Liu-Long] For $n \geq 2$, $\{\rho_{n,\ell}\}$ (hence $L(\mathcal{E}_n, s)$) is potentially automorphic, and automorphic for $n \leq 6$.

Remark. Scholl has shown that, for p large, forms in $S_3(\Gamma_n)$ with p-adically integral Fourier coefficients satisfy a congruence relation with the coefficients of the characteristic polynomial of $\rho_{n,\ell}(\mathrm{Frob}_p)$. If $\rho_{n,\ell}$ were automorphic, then this would be a congruence relation between Fourier coefficients of forms for a non-congruence subgroup with those of a congruence subgroup.