Some Analytic Questions Involving Shintani Zeta Functions

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Binary cubic forms

The lattices of *binary cubic forms* are

$$V(\mathbb{Z}) := \{au^3 + bu^2v + cuv^2 + dv^3 \ : \ a, b, c, d \in \mathbb{Z}\}.$$

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which satisfies

$$\mathrm{Disc}(g \circ f) = (\det g)^2 \mathrm{Disc}(f).$$

The Shintani zeta function

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The *Shintani zeta function* is

$$\xi^{\pm}(s) := \sum_{\substack{x \in \mathrm{GL}_2(\mathbb{Z}) \backslash V(\mathbb{Z}) \\ \pm \mathrm{Disc}(x) > 0}} \frac{1}{|\mathrm{Stab}(x)|} |\mathrm{Disc}(x)|^{-s}.$$

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Arithmetic interpretation: This can also be written

$$\xi^{\pm}(s) := \sum_{R \,:\, \pm \mathrm{Disc}(R) > 0} \frac{1}{|\mathrm{Aut}(R)|} |\mathrm{Disc}(R)|^{-s},$$

where the sum is over *cubic rings*.

The analytic functional equation

Theorem (Shintani)

The Shintani zeta functions converge absolutely for $\Re(s) > 1$. They continue to functions holomorphic in the plane except for simple poles at 1 and $5/6$, and satisfy the functional equation

$$\begin{pmatrix} \xi^+(1-s) \\ \xi^-(1-s) \end{pmatrix} = \Gamma\left(s - \frac{1}{6}\right)\Gamma(s)^2\Gamma\left(s + \frac{1}{6}\right)2^{-1}3^{6s-2}\pi^{-4s} \times \begin{pmatrix} \sin 2\pi s & \sin \pi s \\ 3\sin \pi s & \sin 2\pi s \end{pmatrix} \begin{pmatrix} \widehat{\xi}^+(s) \\ \widehat{\xi}^-(s) \end{pmatrix}.$$

Moreover,

$$\mathrm{Res}_{s=1}\xi^{\pm}(s) = \frac{\pi^2(3 + C^{\pm})}{36}, \quad \mathrm{Res}_{s=\frac{5}{6}}\xi^{\pm}(s) = K^{\pm}\frac{\zeta(1/3)\Gamma(1/3)^3}{4\sqrt{3}\pi}.$$

The algebraic functional equation

Theorem (Ohno-Nakagawa)

We have

$$\begin{pmatrix} \widehat{\xi^+}(s) \\ \widehat{\xi^-}(s) \end{pmatrix} = 27^{-s} \cdot \begin{pmatrix} 0 & 1 \\ 3 & 0 \end{pmatrix} \begin{pmatrix} \xi^+(s) \\ \xi^-(s) \end{pmatrix}. \tag{1}$$

The diagonalized functional equation

Define *diagonalized Shintani zeta functions*

$$\xi^{\mathrm{add}}(s) := 3^{1/2}\xi^{+}(s) + \xi^{-}(s),$$

$$\xi^{\mathrm{sub}}(s) := 3^{1/2}\xi^{+}(s) - \xi^{-}(s),$$

and *completed zeta functions*

$$\Lambda^{\mathrm{add}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{12}\right)\Gamma\left(\frac{s}{2}-\frac{1}{12}\right)\xi^{\mathrm{add}}(s),$$

$$\Lambda^{\mathrm{sub}}(s) := \left(\frac{432}{\pi^4}\right)^{s/2}\Gamma\left(\frac{s}{2}\right)\Gamma\left(\frac{s}{2}+\frac{1}{2}\right)\Gamma\left(\frac{s}{2}+\frac{5}{12}\right)\Gamma\left(\frac{s}{2}+\frac{7}{12}\right)\xi^{\mathrm{sub}}(s).$$

Then, we have

$$\Lambda^{\mathrm{add}}(1-s) = 3\Lambda^{\mathrm{add}}(s),$$

$$\Lambda^{\mathrm{sub}}(1-s) = -3\Lambda^{\mathrm{sub}}(s).$$

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- (Insert your favorite quesiton about zeroes!)

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Theorem

The (cubic) Shintani zeta function cannot be represented as a finite sum of Euler products.

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Question. Should such functional equations be common?

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Improve to $n^{\frac{1}{2}+\epsilon}$ by observing that $h_3(D) \leq h(D) \ll |D|^{\frac{1}{2}+\epsilon}$, further improved to $n^{\frac{1}{3}+\epsilon}$ by Ellenberg-Venkatesh.

Question 5: Invent your own!