Biases in Moments of Satake Parameters and in Zeros near the Central Point in Families of L-Functions

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Bias Conjecture for Elliptic Curves

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Last Summer: Families and Moments

A *one-parameter family* of elliptic curves is given by

$$\mathcal{E} : y^2 \ = \ x^3 + A(T)x + B(T)$$

where $A(T), B(T)$ are polynomials in $\mathbb{Z}[T]$.

- Each specialization of T to an integer t gives an elliptic curve $\mathcal{E}(t)$ over $\mathbb{Q}$.
- The r^{th} *moment* of the Fourier coefficients is

$$A_{r,\mathcal{E}}(p) \ = \ \sum_{t \mod p} a_{\mathcal{E}(t)}(p)^r.$$

Negative Bias in the First Moment

$A_{1,\mathcal{E}}(p)$ and Family Rank (Rosen-Silverman)

If Tate's Conjecture holds for $\mathcal{E}$ then

$$\lim_{X\to\infty} \frac{1}{X} \sum_{p\le X} \frac{A_{1,\mathcal{E}}(p)\log p}{p} \ = \ -\mathrm{rank}(\mathcal{E}/\mathbb{Q}).$$

- By the Prime Number Theorem, $A_{1,\mathcal{E}}(p) = -rp + O(1)$ implies $\mathrm{rank}(\mathcal{E}/\mathbb{Q}) = r$.

Bias Conjecture

Second Moment Asymptotic (Michel)

For families $\mathcal{E}$ with $j(T)$ non-constant, the second moment is

$$A_{2,\mathcal{E}}(p) \;=\; p^2 + O(p^{3/2}).$$

- The lower order terms are of sizes $p^{3/2}$, p, $p^{1/2}$, and 1.

In every family we have studied, we have observed:

Bias Conjecture

The largest lower term in the second moment expansion which does not average to 0 is on average **negative**.

Preliminary Evidence and Patterns

Let $n_{3,2,p}$ equal the number of cube roots of 2 modulo p, and set $c_0(p) = \left[\left(\frac{-3}{p}\right) + \left(\frac{3}{p}\right)\right] p$, $c_1(p) = \left[\sum_{x \bmod p} \left(\frac{x^3-x}{p}\right)\right]^2$, $c_{3/2}(p) = p \sum_{x(p)} \left(\frac{4x^3+1}{p}\right)$.

Family	$A_{1,\mathcal{E}}(p)$	$A_{2,\mathcal{E}}(p)$
$y^2 = x^3 + Sx + T$	0	$p^3 - p^2$
$y^2 = x^3 + 2^4(-3)^3(9T+1)^2$	0	$\begin{cases} 2p^2-2p & p\equiv 2 \bmod 3 \\ 0 & p\equiv 1 \bmod 3 \end{cases}$
$y^2 = x^3 \pm 4(4T+2)x$	0	$\begin{cases} 2p^2-2p & p\equiv 1 \bmod 4 \\ 0 & p\equiv 3 \bmod 4 \end{cases}$
$y^2 = x^3 + (T+1)x^2 + Tx$	0	$p^2 - 2p - 1$
$y^2 = x^3 + x^2 + 2T + 1$	0	$p^2 - 2p - \left(\frac{-3}{p}\right)$
$y^2 = x^3 + Tx^2 + 1$	$-p$	$p^2 - n_{3,2,p}p - 1 + c_{3/2}(p)$
$y^2 = x^3 - T^2x + T^2$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 - T^2x + T^4$	$-2p$	$p^2 - p - c_1(p) - c_0(p)$
$y^2 = x^3 + Tx^2 - (T+3)x + 1$	$-2c_{p,1;4}p$	$p^2 - 4c_{p,1;6}p - 1$

where $c_{p,a;m} = 1$ if $p \equiv a \bmod m$ and otherwise is 0.

Lower order terms and average rank

$$\frac{1}{N}\sum_{t=N}^{2N}\sum_{\gamma_t}\phi\left(\gamma_t\frac{\log R}{2\pi}\right) = \widehat{\phi}(0)+\phi(0)-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p}\widehat{\phi}\left(\frac{\log p}{\log R}\right)a_t(p)$$
$$-\frac{2}{N}\sum_{t=N}^{2N}\sum_{p}\frac{\log p}{\log R}\frac{1}{p^2}\widehat{\phi}\left(\frac{2\log p}{\log R}\right)a_t(p)^2+O\left(\frac{\log\log R}{\log R}\right).$$

- $\phi(x)\geq 0$ gives upper bound average rank.
- Expect big-Oh term $\Omega(1/\log R)$.

Implications for Excess Rank

- Katz-Sarnak's one-level density statistic is used to measure the average rank of curves over a family.
- More curves with rank than expected have been observed, though this excess average rank vanishes in the limit.
- Lower-order biases in the moments of families explain a small fraction of this excess rank phenomenon.

Methods for Obtaining Explicit Formulas

For a family $\mathcal{E}: y^2 = x^3 + A(T)x + B(T)$, we can write

$$a_{\mathcal{E}(t)}(p) \;=\; -\sum_{x \mod p} \left(\frac{x^3 + A(t)x + B(t)}{p}\right)$$

where $\left(\frac{\cdot}{p}\right)$ is the Legendre symbol mod$p$ given by

$$\left(\frac{x}{p}\right) \;=\; \begin{cases} 1 & \text{if } x \text{ is a non-zero square modulo } p \\ 0 & \text{if } x \equiv 0 \bmod p \\ -1 & \text{otherwise.} \end{cases}$$

Lemmas on Legendre Symbols

Linear and Quadratic Legendre Sums

$$\sum_{x \bmod p} \left(\frac{ax+b}{p}\right) = 0 \quad \text{if } p \nmid a$$

$$\sum_{x \bmod p} \left(\frac{ax^2+bx+c}{p}\right) = \begin{cases} -\left(\frac{a}{p}\right) & \text{if } p \nmid b^2-4ac \\ (p-1)\left(\frac{a}{p}\right) & \text{if } p \mid b^2-4ac \end{cases}$$

Average Values of Legendre Symbols

The value of $\left(\frac{x}{p}\right)$ for $x \in \mathbb{Z}$, when averaged over all primes p, is 1 if x is a non-zero square, and 0 otherwise.

Rank 0 Families

Theorem (MMRW'14): Rank 0 Families Obeying the Bias Conjecture

For families of the form $\mathcal{E} : y^2 = x^3 + ax^2 + bx + cT + d$,

$$A_{2,\mathcal{E}}(p) \;=\; p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{a^2-3b}{p}\right)\right).$$

- The average bias in the size p term is -2 or -1, according to whether $a^2 - 3b \in \mathbb{Z}$ is a non-zero square.

Families with Rank

Theorem (MMRW'14): Families with Rank

For families of the form $\mathcal{E} : y^2 = x^3 + aT^2x + bT^2$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - p\left(1 + \left(\frac{-3}{p}\right) + \left(\frac{-3a}{p}\right)\right) - \left(\sum_{x(p)} \left(\frac{x^3+ax}{p}\right)\right)^2.$$

- These include families of rank 0, 1, and 2.
- The average bias in the size p terms is -3 or -2, according to whether $-3a \in \mathbb{Z}$ is a non-zero square.

Families with Rank

Theorem (MMRW'14): Families with Complex Multiplication

For families of the form $\mathcal{E} : y^2 = x^3 + (aT + b)x$,

$$A_{2,\mathcal{E}}(p) \;=\; (p^2 - p)\left(1 + \left(\frac{-1}{p}\right)\right).$$

- The average bias in the size p term is -1.
- The size p^2 term is not constant, but is on average p^2, and an analogous Bias Conjecture holds.

Families with Unusual Distributions of Signs

Theorem (MMRW'14): Families with Unusual Signs

For the family $\mathcal{E} : y^2 = x^3 + Tx^2 - (T+3)x + 1$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - p\left(2 + 2\left(\frac{-3}{p}\right)\right) - 1.$$

- The average bias in the size p term is -2.
- The family has an usual distribution of signs in the functional equations of the corresponding L-functions.

The Size $p^{3/2}$ Term

Theorem (MMRW'14): Families with a Large Error

For families of the form
$\mathcal{E} : y^2 = x^3 + (T + a)x^2 + (bT + b^2 - ab + c)x - bc$,

$$A_{2,\mathcal{E}}(p) \ = \ p^2 - 3p - 1 + p \sum_{x \mod p} \left(\frac{-cx(x+b)(bx-c)}{p} \right)$$

- The size $p^{3/2}$ term is given by an elliptic curve coefficient and is thus on average 0.
- The average bias in the size p term is -3.

General Structure of the Lower Order Terms

The lower order terms appear to always

- have no size $p^{3/2}$ term or a size $p^{3/2}$ term that is on average 0;
- exhibit their negative bias in the size p term;
- be determined by polynomials in p, elliptic curve coefficients, and congruence classes of p (i.e., values of Legendre symbols).

New Families: Work in Progress

- Dirichlet characters of prime level: bias $+1$.
- Holomorphic cusp forms: bias $-1/2$.
- r^{th} Symmetric Power $\mathcal{F}_{r,X,\delta,q}$: bias $+1/48$.

(With Megumi Asada and Eva Fourakis (Williams), Kevin Yang (Harvard).)

Finite Conductor Models at Central Point

With Owen Barrett and Blaine Talbut (Chicago), Gwyn Moreland (Michigan), Nathan Ryan (Bucknell)

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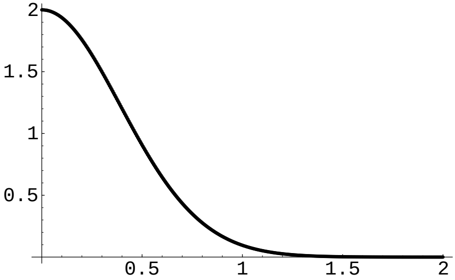
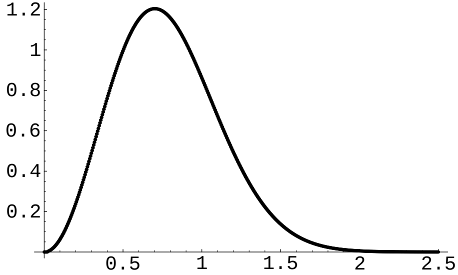
Excised Orthogonal Ensemble joint with Eduardo Dueñez, Duc Khiem Huynh, Jon Keating and Nina Snaith. Numerical experiments ongoing with Nathan Ryan.

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(even)

RMT: Theoretical Results ($N \to \infty$)



1st normalized evalue above 1: SO(odd)

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

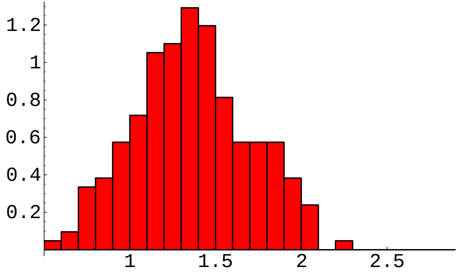


Figure 4a: 209 rank 0 curves from 14 rank 0 families, $\log(\text{cond}) \in [3.26, 9.98]$, median $= 1.35$, mean $= 1.36$

Rank 0 Curves: 1st Norm Zero: 14 One-Param of Rank 0

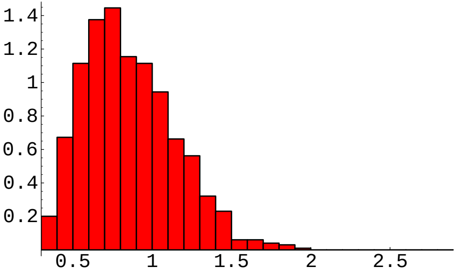


Figure 4b: 996 rank 0 curves from 14 rank 0 families, $\log(\mathrm{cond}) \in [15.00, 16.00]$, median $= .81$, mean $= .86$.

Spacings b/w Norm Zeros: Rank 0 One-Param Families over $\mathbb{Q}(T)$

- All curves have $\log(\mathrm{cond}) \in [15, 16]$;
- $z_j =$ imaginary part of j^{th} normalized zero above the central point;
- 863 rank 0 curves from the 14 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$.

	863 Rank 0 Curves	**701 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.28	1.30	
Mean $z_2 - z_1$	1.30	1.34	-1.60
StDev $z_2 - z_1$	0.49	0.51	
Median $z_3 - z_2$	1.22	1.19	
Mean $z_3 - z_2$	1.24	1.22	0.80
StDev $z_3 - z_2$	0.52	0.47	
Median $z_3 - z_1$	2.54	2.56	
Mean $z_3 - z_1$	2.55	2.56	-0.38
StDev $z_3 - z_1$	0.52	0.52	

Spacings b/w Norm Zeros: Rank 2 one-param families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$;
- 23 rank 4 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	64 Rank 2 Curves	23 Rank 4 Curves	t-Statistic
Median $z_2 - z_1$	1.26	1.27	
Mean $z_2 - z_1$	1.36	1.29	0.59
StDev $z_2 - z_1$	0.50	0.42	
Median $z_3 - z_2$	1.22	1.08	
Mean $z_3 - z_2$	1.29	1.14	1.35
StDev $z_3 - z_2$	0.49	0.35	
Median $z_3 - z_1$	2.66	2.46	
Mean $z_3 - z_1$	2.65	2.43	2.05
StDev $z_3 - z_1$	0.44	0.42	

Rank 2 Curves from Rank 0 & Rank 2 Families over $\mathbb{Q}(T)$

- All curves have $\log(\text{cond}) \in [15, 16]$;
- z_j = imaginary part of the j^{th} norm zero above the central point;
- 701 rank 2 curves from the 21 one-param families of rank 0 over $\mathbb{Q}(T)$;
- 64 rank 2 curves from the 21 one-param families of rank 2 over $\mathbb{Q}(T)$.

	701 Rank 2 Curves	**64 Rank 2 Curves**	**t-Statistic**
Median $z_2 - z_1$	1.30	1.26	
Mean $z_2 - z_1$	1.34	1.36	0.69
StDev $z_2 - z_1$	0.51	0.50	
Median $z_3 - z_2$	1.19	1.22	
Mean $z_3 - z_2$	1.22	1.29	1.39
StDev $z_3 - z_2$	0.47	0.49	
Median $z_3 - z_1$	2.56	2.66	
Mean $z_3 - z_1$	2.56	2.65	1.93
StDev $z_3 - z_1$	0.52	0.44	

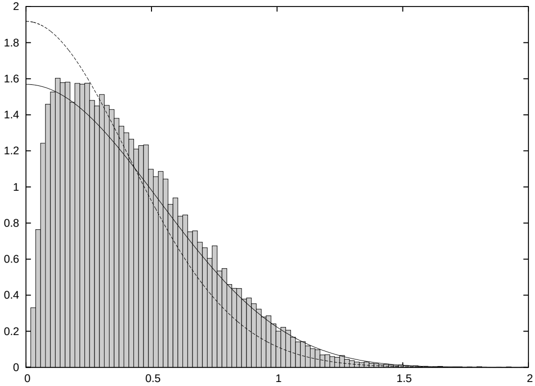
New Model for Finite Conductors

- Replace conductor N with $N_{\text{effective}}$.
  - ⋄ Arithmetic info, predict with L-function Ratios Conj.
  - ⋄ Do the number theory computation.

- Excised Orthogonal Ensembles.
  - ⋄ $L(1/2, E)$ discretized.
  - ⋄ Study matrices in $\mathrm{SO}(2N_{eff})$ with $|\Lambda_A(1)| \geq ce^N$.

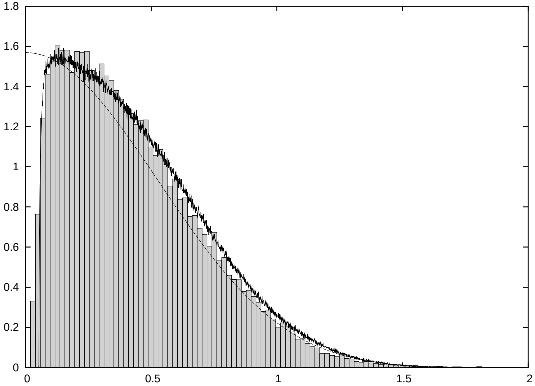
- Painlevé VI differential equation solver.
  - ⋄ Use explicit formulas for densities of Jacobi ensembles.
  - ⋄ Key input: Selberg-Aomoto integral for initial conditions.

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart), lowest eigenvalue of SO(2N) with N_{eff} (solid), standard N_0 (dashed).

Modeling lowest zero of $L_{E_{11}}(s, \chi_d)$ with $0 < d < 400,000$



Lowest zero for $L_{E_{11}}(s, \chi_d)$ (bar chart); lowest eigenvalue of SO(2N): N_{eff} = 2 (solid) with discretisation, and $N_{\mathrm{eff}} = 2.32$ (dashed) without discretisation.

Effective Matrix Size: Families with Unitary Symplectic Monodromy

- *L*-function attached to quadratic Dirichlet character.
  - ◇ $L(\chi, s) \ = \ \prod_{p<\infty} (1 - \chi(p)p^{-s})^{-1}$.
- *L*-function attached to symmetric power.
  - ◇ $L(\mathrm{Sym}^r f, s) \ = \ \prod_{p<\infty} L_p(\mathrm{Sym}^r f, s)$.
- Compute 1-level Density: Study distribution of zeros
  - ◇ $D_{1,\varphi}(\mathcal{F}) = \#\mathcal{F}^{-1} \cdot \sum_{f\in\mathcal{F}} \sum_{\rho_f = 1/2 + i\gamma_f} \varphi(\gamma_f \cdot \frac{\log Q}{2\pi})$

Integral Representation of One-Level Density

We bound conductors of families by a parameter X

- ⋄ For quadratic Dirichlet characters, we have:

Theorem

The One-Level Density is represented by the integral kernel

$$K(\tau) = 1 - \frac{\sin(2\pi\tau)}{2\pi\tau} + \frac{1 - \cos(2\pi\tau)}{\Lambda^{-1}\log X} + O\left(\frac{1}{\log^2 X}\right)$$

for $\Lambda < 0$.

Similarly for the family of quadratic twists of $\mathrm{Sym}^r f$.

Deducing Effective Matrix Size

- Matching with integral kernel of matrix groups.
  - ◇ $\frac{\pi}{N} \cdot K_{1,USp(2N)}(t) = 1 - \frac{\sin(2\pi t)}{2\pi t} + \frac{1-\cos(2\pi t)}{2N} + \cdots$
  - ◇ $\frac{\pi}{N} \cdot K_{1,SO(2N+1)}(t)$, same leading term.
- Note

$$\frac{\pi}{N} \cdot \left(K_{1,\mathrm{SO}(2N+1)} - K_{1,\mathrm{USp}(2(-N))}\right) \sim \frac{1-\cos(2\pi t)}{2N}$$

- Unitary Symplectic Families behave like $\mathrm{SO}(2N+1)$ for bounded X.
- Similarly for quadratic twists of $\mathrm{Sym}^2 f$.

Excised Orthogonal Ensemble

- As before, let $\mathcal{F}$ be those quadratic twists of $L(E,s)$.
- Idea: interpret $L(E,\frac{1}{2}+it)$ as an integral kernel.
- Taylor Series expansion:

$$L(E,s) = L(E,\frac{1}{2}) + L'(E,\frac{1}{2})(s-\frac{1}{2}) + \dots$$

- Goal: match power series coefficients with that of $\text{ch}_H(e^{i\theta})$.
- Amalgamate integral kernels together: attach to $\mathcal{F}$ a product distribution $\prod_{E\in\mathcal{F}} \int_0^\infty L(E,\frac{1}{2}+it)dt$.

Excised Orthogonal Ensemble (continued)

We deduce

Theorem

Let $\mathcal{F}_X$ be those quadratic twists of an elliptic curve $E/\mathbb{Q}$ of conductor $N < X$. If $\sup_n \left(\left| L^{(n)}(E, \frac{1}{2}) - ch^{(n)}(1) \right| \right) < \delta$, then $\left|\left| D_{1,\mathcal{F}_X} - D_{1,\mathcal{M}_{N(X)}} \right|\right|_{L^2} < \varepsilon$.

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Central:

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Bias Conjecture for Elliptic Curves

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Summary of Results

- Dirichlet characters of prime level: bias $+1$.
- Holomorphic cusp forms: bias $-1/2$.
- r^{th} Symmetric Power $\mathcal{F}_{r,X,\delta,q}$: bias $+1/48$.

Dirichlet Family $\mathcal{F}_q$

Definition

Prime $q \in \mathbb{Z}$ and $\mathcal{F}_q \ = \ \{\chi \neq \chi_0(q)\}$ is the family of nontrivial Dirichlet characters of conductor q. The *second moment at p* is

$$M_2(\mathcal{F}_q; p) \ := \ \sum_{\chi \in \mathcal{F}_q} \chi^2(p).$$

Goal: Compute asymptotics for the sum

$$M_{2,X}(\mathcal{F}_q) = \sum_{p<X} M_2(\mathcal{F}_q; p) = \sum_{p<X} \sum_{\chi \in \mathcal{F}_q} \chi^2(p).$$

Results for $\mathcal{F}_q$

Theorem

Family $\mathcal{F}_q$ has positive bias in the second moment of $+1$.

Have $M_2(\mathcal{F}_q; p) := \sum_{\chi \in \mathcal{F}_q} \chi^2(p)$.

From orthogonality relations:

$$M_2(\mathcal{F}_q; p) \ = \ \begin{cases} q-2 & \text{if } p \equiv \pm 1 (q); \\ -1 & \text{if } p \not\equiv \pm 1 (q), \end{cases}$$

Thus

$$\sum_{p<X} M_2(\mathcal{F}_q; p) \ = \ \sum_{\substack{p<X \\ p\equiv\pm1(q)}} (q-2) \ - \ \sum_{\substack{p<X \\ p\not\equiv\pm1(q)}} 1.$$

Main term size $\pi(X)$. □

Cuspidal Newforms

Fix level $q = 1$. For weight k, consider an orthonormal basis $\mathcal{B}_{k,q}(\chi_0)$ of $H_{k,q}(\chi_0)$, the space of holomorphic cusp forms on the surface $\Gamma_0\backslash\mathfrak{h}$ of level k and trivial nebentypus.

Family

$$\mathcal{F}_X := \bigcup_{\substack{k<X \\ k\equiv 0(2)}} \mathcal{B}_{k,q=1}(\chi_0).$$

An Important Tool: Petersson Trace Formula

For any $n, m \geq 1$, we have

$$\frac{\Gamma(k-1)}{(4\pi p)^{k-1}} \sum_{f \in B_{k,q}(\chi_0)} |\lambda_f(p)|^2 = \delta(p,p) + 2\pi i^{-k} \sum_{c \equiv 0(q)} \frac{S_c(p,p)}{c} J_{k-1}\left(\frac{4\pi p}{c}\right)$$

where $\lambda_f(n)$ is the n-th Hecke eigenvalue of f,
$\delta(m,n)$ is Kronecker's delta,
$S_c(m,n)$ is the classical Kloosterman sum, and
$J_{k-1}(t)$ is the k-Bessel function.

Cusp Newform: $\mathcal{F}_{<X}$

We gain asymptotic control over $J_{k-1}(t)$ by averaging over even weights k.

$$M_2(\mathcal{F}_X;p) \;=\; \sum_{k^*<X} M_2(H_{k,1}(\chi_0);p) \;=\; \sum_{k^*<X} \sum_{f\in\mathcal{B}_{k,1}(\chi_0)} |\lambda_f(p)|^2$$

where $\sum_{k^*<X}$ denotes summing over even k.

Theorem

Let $\varphi \in C_0^\infty(\mathbb{R}_{>0})$ be real-valued, and let $X > 1$. Then

$$4\sum_{k\equiv 0(2)} \varphi\left(\frac{k-1}{X}\right) J_{k-1}(t) \;=\; \varphi\left(\frac{t}{X}\right) \;+\; \frac{t}{6X^3}\varphi^{(2)}\left(\frac{t}{X}\right)$$

Cusp Newform: $\mathcal{F}_{<X}$

To handle $S_c(m,n)$, we instead compute

$$M_2\left(\mathcal{F}_X;\delta\right) \;=\; \sum_{p<X^\delta} M_2\left(\mathcal{F}_X;p\right)\cdot\log p.$$

After several substitutions and iterations of integration by parts,

$$M_2(\mathcal{F}_X;\delta) \;=\; \frac{1}{2}X^{1+\delta} - \frac{X^{1+\delta}}{2\log^2 X^\delta} + O\left(\frac{X^{1+\delta}}{\log^3 X^\delta}\right)$$

yields a bias of $-1/2$.

Varying the Level: $\mathcal{F}_X; \delta; \epsilon$

Can also vary the level:

$$\begin{aligned} M_2(\mathcal{F}_X;\delta;\varepsilon) &= \sum_{q<X^\varepsilon} M_2(\mathcal{F}_{q,X};\delta) \\ &= \sum_{q<X^\varepsilon} \sum_{p<X^\delta} \sum_{k*<X} \sum_{f\in B_{k,q}(\chi_0)} |\lambda_f(p)|^2 \cdot \log p \\ &= \frac{1}{2}X^{1+\delta+\varepsilon} - \frac{X^{1+\delta+\varepsilon}}{2\log^2 X^\delta} + O\left(\frac{X^{1+\delta+\varepsilon}}{\log^3 X^\delta}\right). \end{aligned}$$

Symmetric Lift Family

Fix a square-free level q and study for $\delta > 0$

$$\mathcal{F}_{r,X,\delta,q} \ = \bigcup_{k<X^\delta} \mathrm{Sym}^{\mathrm{r}}\left[\mathrm{H}^*_{\mathrm{k,q}}(\chi_0)\right].$$

Second moment: for $\varepsilon > 0$:

$$M_{2,\varepsilon}(\mathcal{F}_{r,X,\delta,q}) \ = \ \frac{1}{\varphi(q)} \sum_{p<X^\varepsilon} \sum_{k<X^\delta} \left(\sum_{f\in H^*_{k,q}(\chi_0)} \lambda^2_{\mathrm{Sym^r f}}(p) \right),$$

find bias of $+1/48$ in

$$M_{2,\varepsilon}(\mathcal{F}_{r,X,\delta}) \ = \ \lim_{\substack{q\to\infty \\ q \ \mathrm{sq-free}}} M_{2,\varepsilon}(\mathcal{F}_{r,X,\delta,q}).$$