Computing central values of twisted L-functions of higher degree

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Computational challenges

We want to compute values of L-functions on the critical line and often we face

- Not knowing the functional equation.
- Not having enough Dirichlet series coefficients.
- Having a fair number of coefficients but having huge conductor.

Huge conductors

Experiment: Fix an L-function $L(s, \pi)$ with, for example, sign $+1$. Consider the set of fundamental discriminants D and quadratic characters χ_D so that $L(s, \pi \otimes \chi_D)$ also has sign $+1$. For $D < 400000$ find

1. the D for which $L(1/2, \pi \otimes \chi_D)$ vanish;
2. the lowest lying zero above $s = 1/2$.

- With S. Miller and O. Barrett, carrying out these computations for particular classical modular forms to check some RMT predictions.
- The conductor of $L(s, \pi \otimes \chi_D)$ grows very quickly with $|D|$.

Random Matrix Theory predictions

Question: *Given a holomorphic newform f with integral coefficients and associated L-function $L(s, f)$, for how many fundamental discriminants D with $|D| \leq x$, does $L(s, f \otimes \chi_D)$, the L-function twisted by the real, primitive, Dirichlet character associated with the discriminant D, vanish at the center of the critical strip to order at least 2?*

- if we consider the elliptic curve E attached to f, RMT predicts

$$V_E(x) \sim b_E x^{3/4} (\log x)^{e_E} .$$

- How does one compute enough data to verify a conjecture of this kind? Especially, the $\log x$ term.
- How do we even know that $L(1/2, f \otimes \chi_D) = 0$?

Waldspurger

- Suppose we compute $L(1/2, f \otimes \chi_D)$ and get $0.00003234\ldots$? Is this zero?
- Suppose we want to compute $L(1/2, f \otimes \chi_D)$ to 32 decimal places. How many coefficients do we need if $D \approx 400000$?
- Waldspurger's Formula (also Gross-Zagier) gives us a way to do both of these at once: for every f there is a half-integer weight modular form g_f with coefficients $c_{g_f}(n)$ so that

$$L(1/2, f \otimes \chi_D) = k_f c_{g_f}(|D|)^2/|D|^{k-1/2}.$$

Example: Used by Hart, Tornaría and Watkins to find all congruent numbers up to 10^{12}.

Degree 4 L-functions

- "Smallest" L-functions attached to Hilbert and Siegel modular forms have degree 4.
- RMT predictions for the number of vanishings of central values depend only on the Γ-factors.
- In particular, if

$$\Lambda(s) = (D^2\sqrt{q})^s)\Gamma(s+a)\Gamma(s+b)L(s,\chi_D) = \pm\Lambda(1-s),$$

  then the number of vanishings for D up to X should be about

$$X^{1-(a+b)/2}.$$

- In particular, this tells us that we need to look at L-functions of Hilbert and Siegel modular forms of small weight.

L-functions attached to Hilbert modular forms

Conjecture

Let $f \in S_k(\mathfrak{N})$ be a Hilbert newform of odd squarefree level $\mathfrak{N}$ such that k satisfies the parity condition. Then there exists a modular form $g(z) = \sum_{\mu \in (\mathfrak{D}^{-1})_+} c_\mu q^{\mathrm{Tr}(\mu z)} \in S_{(k+1)/2}(4\mathfrak{N})$ such that for all permitted $D \in \mathcal{D}(\mathbb{Z}_F)$, we have

$$L(f, 1/2, \chi_D) = \kappa_f \frac{c_{|D|}(g)^2}{\prod_{i=1}^{n} \sqrt{|v_i(D)|}^{k_i - 1}},$$

where $\kappa_f \neq 0$ is independent of D. In the case of parallel weight k the denominator in the right hand side is just $\sqrt{\mathrm{N}(D)}^{k-1}$.

Conjecture

There exist $b_f, C_f \geq 0$ depending on f such that as $X \to \infty$, we have

$$\mathcal{N}_f(\mathbb{Z}_F; X) \sim C_f X^{1-(k-1)/4} (\log X)^{b_f}.$$

Conjecture

There exist $b_{f,\mathbb{Z}}, C_{f,\mathbb{Z}} \geq 0$ depending on f such that as $X \to \infty$, we have

$$\mathcal{N}_f(\mathbb{Z}; X) \sim C_{f,\mathbb{Z}} X^{1-n(k-1)/4} (\log X)^{b_{f,\mathbb{Z}}}.$$

Experimental results

- Developed algorithm to compute half-integer weight Hilbert modular forms.
- Computed twists up to 1.5×10^8 for one Hilbert modular form.
- Still not enough to verify the RMT predictions (the log term, in particular).

L-functions attached to Siegel modular forms

For a fundamental discriminant $D < 0$ coprime to the level, Böcherer's Conjecture states:

$$L(F, 1/2, \chi_D) = C_F |D|^{1-k} A(D)^2$$

where F is a Siegel modular form of weight k, $C_F > 0$ is a constant that only depends on F, and $A(D)$ is an average of the coefficients of F of discriminant D.

Some context

- A theorem of A. Saha states that a weak version of the conjecture implies multiplicity one for Siegel modular forms of level 1.
- It's a generalization of Waldspurger's formula relating central values of elliptic curve L-functions to sums of coefficients of half-integer weight modular forms.
- Some how it's better than Waldspurger: it says that a SMF knows its own central values whereas in Waldspurger, we need this half-integral weight form on the RHS.
- In general, computing coefficients of Siegel modular forms is much easier than computing their Hecke eigenvalues (and therefore their L-functions). So this formula would provide a computationally feasible way to compute lots of central values.

The state of the art

- Böcherer originally proved it for Siegel modular forms that are Saito-Kurokawa lifts.
- Kohnen and Kuss verified the conjecture numerically for the first few rational Siegel modular eigenforms that are not lifts (these are in weight 20-26) for only a few fundamental discriminants.
- Raum recently verified the conjecture numerically for nonrational Siegel modular eigenforms that are not lifts for a few more fundamental discriminants.
- Böcherer and Schulze-Pillot formulated a conjecture for Siegel modular forms with level > 1 and proved it when the form is a Yoshida lift.

Suppose we are given a paramodular form $F \in S^k(\Gamma^{\text{para}}[p])$ so that for all $n \in \mathbb{Z}$, $F|T(n) = \lambda_{F,n} F = \lambda_n F$ where $T(n)$ is the nth Hecke operator. Then we can define the spin L-series by the Euler product

$$L(F,s) := \prod_{q \text{ prime}} L_q(q^{-s-k+3/2})^{-1},$$

where the local Euler factors are given by

$$L_q(X) := 1 - \lambda_q X + (\lambda_q^2 - \lambda_{q^2} - q^{2k-4})X^2 - \lambda_q q^{2k-3} X^3 + q^{4k-6} X^4$$

for $q \neq p$, and $L_p(X)$ has a similar formula.

We define

$$A_F(D) := \sum_{\{T>0 \,:\, \text{disc } T = D\}/\hat{\Gamma}_0(p)} \frac{a(T;F)}{\varepsilon(T)}$$

where $\varepsilon(T) := \#\{U \in \hat{\Gamma}_0(p) : T[U] = T\}$.

Conjecture (Paramodular Böcherer's Conjecture, I)

Suppose $F \in S^k(\Gamma^{\text{para}}[p])^+$. Then, for fundamental discriminants $D < 0$ we have

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

where C_F is a positive constant that depends only on F, and $\star = 1$ when $p \nmid D$, and $\star = 2$ when $p \mid D$.

Theorem (R., Tornaría)

Let $F = Grit(f) \in S^k(\Gamma^{para}[p])^+$ *where* p *is prime and* f *is a Hecke eigenform of degree 1, level* p *and weight* $2k-2$*. Then there exists a constant* $C_F > 0$ *so that*

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

for $D < 0$ *a fundamental discriminant, and* $\star = 1$ *when* $p \nmid D$*, and* $\star = 2$ *when* $p \mid D$*.*

The idea of the proof is to combine four ingredients:

- the factorization of the L-function of the Gritsenko lift as given by Ralf Schmidt,
- Dirichlet's class number formula,
- the explicit description of the Fourier coefficients of the Gritsenko lift and
- Waldspurger's theorem.

Theorem (R., Tornaría)

Let $F \in S^2(\Gamma^{para}[p])^+$ where $p < 600$ is prime. Then, numerically, there exists a constant $C_F > 0$ so that

$$L(F, 1/2, \chi_D) = \star\, C_F |D|^{1-k} A(D)^2$$

for $-200 \leq D < 0$ a fundamental discriminant, and $\star = 1$ when $p \nmid D$, and $\star = 2$ when $p \mid D$.

Results of Cris Poor and Dave Yuen:

- Determine what levels of weight 2 paramodular cuspforms have Hecke eigenforms that are not Gritsenko lifts.
- Provide Fourier coefficients (up to discriminant 2500) for all paramodular forms of prime level up to 600 that are not Gritsenko – not enough to compute central values of twists.

Brumer and Kramer formulated the following conjecture:

Conjecture (Paramodular Conjecture)

Let p be a prime. There is a bijection between lines of Hecke eigenforms $F \in S^2(\Gamma^{\mathrm{para}}[p])$ that have rational eigenvalues and are not Gritsenko lifts and isogeny classes of rational abelian surfaces $\mathcal{A}$ of conductor p. In this correspondence we have that

$$L(\mathcal{A}, s, \text{Hasse-Weil}) = L(F, s).$$

We remark that it is merely expected that the two L-series mentioned above have an analytic continuation and satisfy a functional equation.

In our computations we assume the Paramodular conjecture for these curves:

p	ϵ	C
277	$+$	$y^2 + y = x^5 - 2x^3 + 2x^2 - x$
349	$+$	$y^2 + y = -x^5 - 2x^4 - x^3 + x^2 + x$
389	$+$	$y^2 + xy = -x^5 - 3x^4 - 4x^3 - 3x^2 - x$
461	$+$	$y^2 + y = -2x^6 + 3x^5 - 3x^3 + x$
523	$+$	$y^2 + xy = -x^5 + 4x^4 - 5x^3 + x^2 + x$
587	$+$	$y^2 = -3x^6 + 18x^4 + 6x^3 + 9x^2 - 54x + 57$
587	-	$y^2 + \left(x^3 + x + 1\right) y = -x^3 - x^2$

The Selberg data we use are:

- $L^*(F,s) = \left(\frac{\sqrt{p}}{4\pi^2}\right)^s \Gamma(s+1/2)\Gamma(s+1/2)L(F,s)$.
- conjecturally $L^*(F,s) = \epsilon\, L^*(F,1-s)$ when $F \in S^2(\Gamma^{\mathrm{para}}[p])^{\epsilon}$.
- we use Mike Rubinstein's `lcalc` to compute the central values using this Selberg data and Sage code we wrote to compute the coefficients of the Hasse-Weil L-function

D	$A(D;F_{277})$	$\frac{L(F_{277},1/2,\chi_D)}{C_{277}}\,\lvert D\rvert$	D	$A(D;F_{277})$	$\frac{L(F_{277},1/2,\chi_D)}{C_{277}}\,\lvert D\rvert$
-3	-1	1.000000	-83	6	36.000000
-4	-1	1.000000	-84	1	1.000000
-7	-1	1.000000	-87	-3	9.000000
-19	-2	4.000000	-88	-2	4.000000
-23	0	-0.000000	-91	-1	1.000000
-39	1	1.000000	-116	3	9.000000
-40	-6	36.000000	-120	-2	4.000000
-47	0	0.000000	-123	-1	1.000000
-52	5	25.000000	-131	-10	100.000000
-55	-2	4.000000	-136	-6	36.000000
-59	3	9.000000	-155	-10	100.000000
-67	-8	64.000000	-164	-5	25.000000
-71	2	4.000000	-187	8	64.000001
-79	0	0.000000	-191	2	3.999999

Two surprises

Suppose $F \in S^k(\Gamma^{\text{para}}[p])^-$, and let $D < 0$ be a fundamental discriminant.

- When $\left(\frac{D}{p}\right) = +1$, the Conjecture holds trivially. Indeed, note that for such F the sign of the functional equation is -1 and so the central critical value $L(F, s, \chi_D)$ is zero. On the other hand, $A(D)$ can be shown to be zero using the Twin map defined by Poor and Yuen.
- On the other hand, the formula of Conjecture 4 fails to hold in case $\left(\frac{D}{p}\right) = -1$. Since $A(D)$ is an empty sum for this type of discriminant, the right hand side of the formula vanishes trivially. However, the left hand side is still an interesting central value, not necessarily vanishing.

Two surprises

Let $L_D := L(F_{587}^-, 1/2, \chi_D)\,|D|$. This table shows fundamental discriminants for which $\left(\frac{D}{587}\right) = -1$. The obvious thing to notice is that the numbers in the table appear to be squares and so the natural question to ask is: squares of what?

D	-4	-7	-31	-40	-43	-47
L_D/L_{-3}	1.0	1.0	4.0	9.0	144.0	1.0

Two surprises

Up to now we have only considered twists by imaginary quadratic characters; namely $\chi_D = \left(\frac{\cdot}{D}\right)$ for $D < 0$. What if we consider positive D?

- Since

$$A(D) = A_F(D) := \frac{1}{2} \sum_{\{T>0:\mathrm{disc}\, T=D\}/\hat{\Gamma}_0(p)} \frac{a(T;F)}{\varepsilon(T)}$$

  we see that for $D > 0$ the sum is empty. And so Böcherer's Conjecture shouldn't make sense.

Two surprises

Let $L_D := L(F_{277}, 1/2, \chi_D)\,|D|$ and $\left(\frac{D}{277}\right) = +1$. Again, these seem to be squares, but squares of what?

D	12	13	21	28	29	40
L_D/L_1	225.0	225.0	225.0	225.0	2025.0	900.0

A new conjecture

Conjecture

Let N be squarefree. Suppose $F \in S_k^{\mathrm{new}}(\Gamma^{\mathrm{para}}[N])$ is a Hecke eigenform and not a Gritsenko lift. Let ℓ and d be fundamental discriminants such that $\ell d < 0$ and such that ℓd is a square modulo $4N$. Then

$$B_{\ell,F}(\ell d)^2 = k_F \cdot \left\{ 2^{\nu_N(\ell)} L(F, 1/2, \chi_\ell) \, |\ell|^{k-1} \right\} \cdot \left\{ 2^{\nu_N(d)} L(F, 1/2, \chi_d) \, |d|^{k-1} \right\}$$

for some positive constant k_F independent of ℓ and d.

A new conjecture

Fix ρ such that $\rho^2 \equiv \ell d \pmod{4N}$. Then

$$B_{\ell,F}(\ell d) = \left| 2^{\nu_N(\gcd(\ell,d))} \cdot \sum \psi_\ell(T) \frac{a(T;F)}{\varepsilon(T)} \right|$$

where the sum is over $\{T = [Nm, r, n] > 0 : \operatorname{disc} T = \ell d,\ r \equiv \rho \pmod{2N}\}/\Gamma_0(N)$ and where $\psi_\ell(T)$ is the genus character corresponding to $\ell \mid \operatorname{disc} T$. This is independent of the choice of ρ.

- Essentially, $B_{\ell,F}(\ell d)$ is the same sum as $A_F(\ell d)$, but appropriately twisted by the genus character ψ_ℓ.

Current work: New way to compute a lot of Fourier coefficients

- Joint with Rupert, Sirolli and Tornaría.
- Identify as many of the first Fourier Jacobi coefficients of the form we want to compute as possible using existing data. Identify the Jacobi forms using the modular symbols method to compute Jacobi forms. Use existing techniques to compute a large of coefficients of those Jacobi forms.
- Bootstrap from here by using relations between the Fourier coefficients of Siegel forms and relations between the Fourier Jacboi coefficients of Siegel forms.