# Investigating the vertical distribution of zeros of L-functions

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Computational Aspects of *L*-functions

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Let $\rho = \beta + i\gamma$ denote a nontrivial zero of $\zeta(s)$, and consider the sequence of ordinates of zeros in the critical strip:

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the average size of $\gamma_{n+1} - \gamma_n$ is $\dfrac{2\pi}{\log(\gamma_n)}$.

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**CONJECTURE**

*Gaps between consecutive zeros of $\zeta(s)$ that are arbitrarily small/large, relative to the average gap size, appear infinitely often.*

# What is known? (assuming RH)

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# Gaps between zeros of other $L$-functions

- For large gaps, we consider the following degree 2 $L$-functions:
    - $\zeta_K(s)$ – the Dedekind zeta-function of a quadratic number field $K$ with discriminant $d$
    - $L(s,f)$ – an automorphic $L$-function on GL(2) over $\mathbb{Q}$, where $f$ is either a primitive holomorphic cusp form or a primitive Maass cusp form

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- For small gaps, we consider primitive $L$-functions from the Selberg Class.

As in the case of $\zeta(s)$, we expect that

**Conjecture**

*Gaps between consecutive zeros that are arbitrarily large, relative to the average gap size, appear infinitely often for both $\zeta_K(s)$ and $L(s,f)$.*

# Large gaps between zeros of $\zeta_K(s)$, $L(s,f)$

**Theorem (T., 2014)**

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**Theorem (Barrett, McDonald, Miller, Ryan, T., Winsor, 2015)**

*Assuming GRH for $L(s,f)$, we have $\lambda_f \geq 1.732$.*

These results can be stated unconditionally if we restrict our attention to zeros on the critical line.

### Wirtinger's Inequality

*Let* $g : [a, b] \to \mathbb{C}$ *be continuously differentiable and suppose that* $g(a) = g(b) = 0$. *Then*

$$\int_a^b |g(t)|^2 dt \leq \left(\frac{b-a}{\pi}\right)^2 \int_a^b |g'(t)|^2 dt.$$

# Large Gaps - Hall's Method (modified by Bredberg)

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By understanding the mean-values of $g(t)$ and $g'(t)$, we can obtain a lower bound on gaps between zeros of $g(t)$.

# Idea of argument for $\zeta_K(s)$

Let

$$g(t) := \exp\left(i\nu\mathcal{L}t\right)\zeta_K(\tfrac{1}{2}+it)M(\tfrac{1}{2}+it),$$

where $\nu$ is a real constant that will be chosen later, $\mathcal{L} \sim \log(\sqrt{d}T)$ and $M(s)$ is an amplifier of the form

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$$M(s) = \sum_{h_1h_2 \le y} \frac{d_r(h_1)d_r(h_2)\chi_d(h_2)P[h_1h_2]}{(h_1h_2)^s}$$

where $y = T^\theta, 0 < \theta < 1/4$, and $d_r(h)$ denotes the coefficients of $\zeta(s)^r$.

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Here

$$P[h] = P\left(\frac{\log y/h}{\log y}\right)$$

for $1 \leq h \leq y$ and $P(x)$ is a polynomial.

# Idea of argument for $\zeta_K(s)$

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By our assumption,

$$t_{n+1} - t_n \leq (1 + o(1))\frac{\kappa\pi}{\mathcal{L}_d}.$$

By Wirtinger's Inequality,

$$\int_{t_n}^{t_{n+1}} |g(t)|^2 dt \leq (1 + o(1)) \frac{\kappa^2}{\mathcal{L}^2} \int_{t_n}^{t_{n+1}} |g'(t)|^2 dt.$$

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Summing for zeros between height $T$ and $2T$, we have

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Therefore, if

$$\frac{\mathcal{L}^2}{\kappa^2}\frac{\int_T^{2T} |g(t)|^2 dt}{\int_T^{2T} |g'(t)|^2 dt} > 1,$$

we may conclude that

$$\lambda_K > \kappa.$$

# Mean-values for the case $\zeta_K(1/2+it)$

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Theorem (Bui, Heap, T., 2014 (preprint))

*We have*

$$\int_T^{2T} |g(t)|^2 \, dt \sim \frac{C_r(0)\mathcal{L}_d^2}{(2r^2-1)!((r-1)!)^4} T + O(T\mathcal{L}_d^{2r^2+4r+1})$$

*and*

$$\int_T^{2T} |g'(t)|^2 \, dt \sim \frac{C_r(1)\mathcal{L}_d^4}{(2r^2-1)!((r-1)!)^4} T + O(T\mathcal{L}_d^{2r^2+4r+3})$$

*as* $T \to \infty$, *where* $C_r(0), C_r(1)$ *are constants depending on the coefficients of* $\zeta_K(s)^r$ *and are given explicitly.*

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The choice of $\theta = 1/4, \nu = 1.2773, r = 1$, and

$$P(x) = 1 - 10.8998x + 28.9444x^2 - 22.1343x^3 + 0.6148x^4$$

allows us to conclude that

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**Theorem (Bui, Heap, T., 2014) (preprint)**

*Assuming GRH, we have $\lambda_K \geq 2.866$. That is, there are infinitely many pairs of consecutive zeros of $\zeta_K(s)$ that are more than 2.866 times the average spacing apart.*

# Remark - Length of the amplifier

$$M(s) = \sum_{h_1 h_2 \leq T^\theta} \frac{d_r(h_1) d_r(h_2) \chi_d(h_2) P[h_1 h_2]}{(h_1 h_2)^s}, \qquad 0 < \theta < 1/4$$

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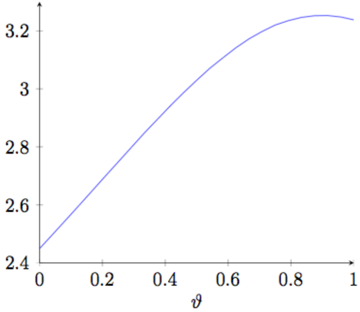


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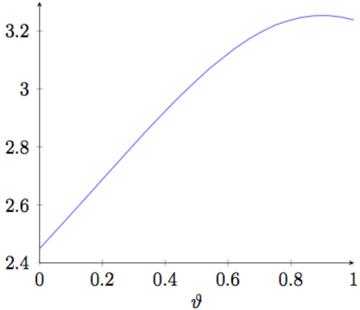


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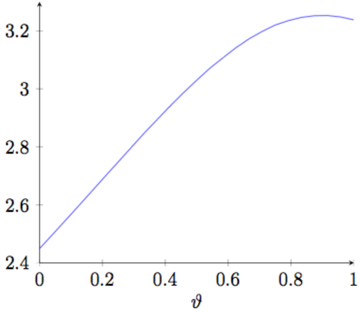


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- The function is not strictly increasing as $\theta \to 1$, (which is widely believed to be the largest value in the range of validity for twisted moment results).
- This phenomenon has also been observed by Bredberg and Bui when studying large gaps between zeros of $\zeta(s)$.

# Remark - higher moments?

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The basic point is that the coefficients in the denominator of the ratio

$$\frac{\int_T^{2T} |g(t)|dt}{\int_T^{2T} |g'(t)|dt}$$

can often be larger than that of the numerator when one considers higher moments.

**Theorem** (Barett, McDonald, Miller, Ryan, T., Winsor, 2015)

*Let $L \in \mathcal{S}$ be primitive of degree $m_L$. Assume GRH and Hypothesis A. Then there is a computable nontrivial upper bound on $\mu_L$ depending on $m_L$. In particular,*

| $m_L$ | upper bound for $\mu_L$ |
|---|---|
| 1 | 0.606894 |
| 2 | 0.822897 |
| 3 | 0.905604 |
| 4 | 0.942914 |
| 5 | 0.962190 |
| ⋮ | ⋮ |

*where the nontrivial upper bounds for $\mu_L$ approach 1 as $m_L$ increases.*

The case $m_L = 1$ has previously been shown by Carneiro, Chandee, Littmann, and Milinovich in 2014.

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- Murty and Perelli proved a general pair correlation result for all primitive $L$-functions in the Selberg Class for restricted support inversely proportional to the degree of the function, assuming the Generalized Riemann Hypothesis and their *Hypothesis A*.
- Hypothesis A is a mild assumption concerning the correlation of the coefficients of $L$-functions at primes and prime powers.