L-functions of degree 4 and weight 1

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joint work with
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Computational Aspects of L-functions
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Goal: To make a catalogue of all L-functions.

In particular:

degree 4, weight 1, rational integer coefficients

(arithmetic normalisation)

Restrictive, but not too restrictive:

- products of elliptic curves over $\mathbb{Q}$
- genus 2 curves over $\mathbb{Q}$
- elliptic curves over quadratic fields $\mathbb{Q}(\sqrt{d})$
- Siegel modular forms
- Hilbert modular forms
- Bianchi modular forms
- Abelian surfaces over $\mathbb{Q}$

L-functions:

degree 4, weight 1, rational integer coefficients

- Dirichlet series $$L(s) = \sum_{n=1}^{\infty} \frac{A_n/\sqrt{n}}{n^s} \qquad A_n \in \mathbb{Z}$$
- Functional equation
$$\Lambda(s) := N^{s/2}\, \Gamma_{\mathbb{C}}(s+\tfrac{1}{2})^2\, L(s) \;=\; \pm\Lambda(1-s)$$
- Euler product $L(s) = \prod_p F_p(p^{-s})^{-1}$, where
$$F_p(z) = G_p(z/\sqrt{p}) \text{ with } G_p(z) \in \mathbb{Z}[z]$$

  Furthermore
  - $F_p(0) = 1$
  - If $p \nmid N$ then $F_p(z)$ has degree 4.
    Also: all roots of $F_p(z)$ lie on $|z| = 1$.
  - If $p | N$ then $F_p(z)$ has degree ≤ 3.
    Also: each root of $F_p(z)$ lies on $|z| = p^{m/2}$
    for some $m \in \{0, 1, 2, 3\}$.

There are only finitely many choices for F_p for each p.

Table: Number of possible local factors

prime, p	good	bad
2	35	26
3	63	32
5	129	38
7	207	44

The Method:

For a given N and $\varepsilon = \pm 1$, everything about the L-function is known *except the coefficients.*

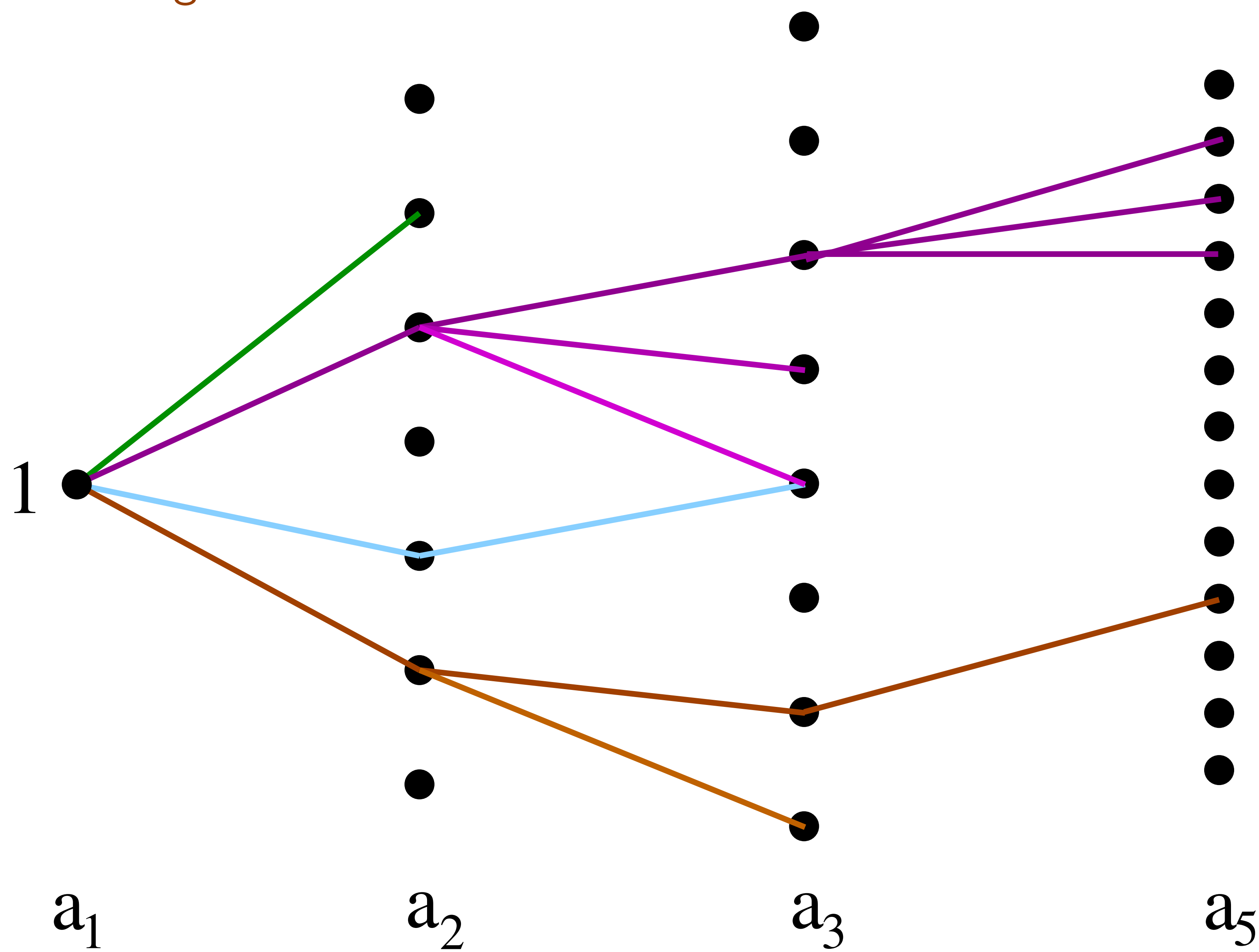
$\Longrightarrow$ find coefficients such that the functional equation is true

OR

prove that the L-function cannot exist.

$\Longrightarrow$ This involves searching a tree.

Searching a tree:



Preliminary results

For degree 4, weight 1 L-functions with rational integer coefficients:

$(N, \varepsilon) \Longrightarrow$ specific functional equation
$+$ the approx. F.E.
$\Longrightarrow$ equation in 'L-function' coefficients

Search:

N: ≤ 680

ε : ± 1

results:

Table of L-functions

Some questions

1. What genus 2 curve or abelian surface has conductor 550?
   (We can tell you the first 200 coefficients.)

2. Why are the first several L-functions non-primitive?

3. Will the primitive L-functions eventually be ‘most’ of them?