L functions in Pari/GP

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A new section

```
gp> ?6
lfun                lfunabelianrelinit lfunan
lfunartin           lfuncheckfeq       lfunconductor
lfuncreate          lfundiv            lfunetaquo
lfunhardy           lfuninit           lfunlambda
lfunmfpeters        lfunmfspec         lfunmul
lfunorderzero       lfunqf             lfunrootres
lfunsymsq           lfunsymsqspec      lfuntheta
lfunthetainit       lfunzeros
```

`lfunblabla`(L,args)

L being either a L-function description, or some precomputation data.

Values, plots, zeros

Values

```
lfun(1,2)
lfun(1,1)
lfun(1,-1)*12
```

Precomputations for a domain $z + [-x, x] + i[-y, y]$

```
pre=lfuninit(1,[.5,0,50]);
ploth(t=-50,50,lfunhardy(pre,t))
```

Zeros

```
lfunzeros(1,50)
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions

```
lfunzeros(x^2+5,20)
lfunzeros(x,20)
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
  - kronecker symbols $\left(\frac{D}{\cdot}\right)$

```
lfunzeros(-3,10)
lfunzeros(5,10)
```

  - character χ mod $N : k \mapsto \zeta_m^{e_k \bmod N}$

```
lfun([6,[0,1,2,0,5,4,0,3,0]],1)
lfunzeros([6,[0,1,2,0,5,4,0,3,0]],10)
```

  - a Hecke character over $\mathbb{Q}$

```
Q=bnfinit(x); ZN=bnrinit(Q,300,1); ZN.cyc
lfuntheta([ZN,[8,1]],1)
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions

```
K=bnfinit(x^2+1); H=bnrinit(K,3,1); H.cyc
lfun([H,[1]],1)
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions
- zeta of quadratic form

```
L=lfunqf([1,0;0,1])
lfun(L,1)
4*lfun(1,1)*lfun(-4,1)
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions
- zeta of quadratic form
- elliptic curves

```
lfunzeros(ellinit("389a1"),10)
lfunorderzero(ellinit("234446a1"))
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions
- zeta of quadratic form
- elliptic curves
- Eta products $f(z) = \prod_{d|N} \eta(dz)^{m_d}$

```
L=lfunetaquo([1,1;3,1;5,1;15,1])
L=lfunetaquo(Mat(1,24))
```

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions
- zeta of quadratic form
- elliptic curves
- Eta products $f(z) = \prod_{d|N} \eta(dz)^{m_d}$
- Artin representations

Classical L functions

Automatic constructions for

- Dedekind zeta functions
- Dirichlet L functions
- Hecke L functions
- zeta of quadratic form
- elliptic curves
- Eta products $f(z) = \prod_{d|N} \eta(dz)^{m_d}$
- Artin representations
- modular forms to come

Operations

Multiplication, division

```
L = lfundiv(lfuncreate(x^2+1),lfuncreate(x))
lfun(L,1)
lfun(-4,1)
```

Symmetric square

```
L = lfunsymsq(ellinit("11a1"))
```

Custom L function

- $L(s) = \sum_{n \geqslant 1} a_n n^{-s}$
- $\gamma(s) = N^{\frac{s}{2}} \prod_{j=1}^{d} \Gamma_{\mathbb{R}}(s + \lambda_i)$
- $\Lambda(s) = L(s)\gamma(s) = \epsilon \Lambda^*(k - s)$

`lfuncreate([`a`,`a^*`,[`λ_1`,...` λ_d`],`k`,`N`,`ϵ`])`

Custom L function

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`lfuncreate([`a`,`a^*`,[`λ_1`,...` λ_d`],`k`,`N`,`ϵ`])`

```
lfuncreate(1)
[[Vecsmall([1]), 1], 0, [0], 1, 1, 1, 1]
lfunetaquo(Mat([1,24]))
[[Vecsmall([7]), Mat([1, 24])], 0, [0, 1], 12, 1, 1]
```

Custom L function

lfuncreate([a,a^*,[λ_1,... λ_d],k,N,ϵ])

```
E=ellinit("37a1");
L=lfuncreate([n->ellan(E,n),0,[0,1],2,37,-1])
lfuncheckfeq(L)
```

```
Fp(p,d) = polsympow(1-ellap(E,p)*x+p*x^2,3)
L=lfuncreate([(p,d)->polsympow(1-ellap(E,p)*x+p*x^2,3),
  0,[-1,0,0,1],4,37^3,-1])
lfuncheckfeq(L)
L=lfuncreate([[(p,d)->polsympow(1-ellap(E,p)*x+p*x^2,3),
  [37,1+x]],0,[-1,0,0,1],4,37^3,-1])
lfuncheckfeq(L)
```

What's behind

- Fourier transform on the critical line

$$f(t) = \Lambda(\frac{w+1}{2} + it), F(x) = \int_{\mathbb{R}} f(t) e^{-2i\pi xt}\,\mathrm{d}t$$

$$F(x) = \sum_{n \geqslant 1} a_n K(ne^x),\ K(t) \sim Ce^{-d\pi t^{\frac{2}{d}}}$$

- Evaluation of inverse Mellin transform $K(t)$ (improved Dokchitser's method)
- Recover $f(t)$ by Poisson summation + functional equation (Booker's method)

$$f(t) = h \sum_k F(kh)(e^{iht})^k + O(e^{-D})$$

Todo

- save a t^1 factor in complexity (exponential smooth factor = shift integration line)
- automatic guesses for missing data
- infinite order Hecke characters
- modular forms
- symmetric powers
- hyperelliptic curves

Work In Progress

Automatically build L functions from the analytic functional equation. Here show number of hits at each level in the search tree.

```
?lfunbuild

for(N=90,100, print(N," -> ",
  lfunbuild([[],0,[0,1],1,N,1],61,[400,2],0)))
90 -> [1, 3, 9, 29, 7, 4, 4, 12, 3, 3, 3, 3, 3, 9, 9, 3, 3, 3]
91 -> [1, 5, 27, 100, 7, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
92 -> [1, 3, 19, 77, 16, 4, 4, 6, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1]
93 -> [1, 5, 14, 56, 8, 5, 5, 6, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
94 -> [1, 3, 20, 83, 21, 6, 5, 5, 1, 1, 1, 1, 1, 2, 1, 1, 1, 1]
95 -> [1, 5, 28, 82, 15, 4, 3, 3, 0, 0, 0, 0, 0, 0, 0, 0, 0, 0]
96 -> [1, 3, 9, 50, 8, 3, 4, 6, 3, 2, 2, 2, 2, 4, 2, 2, 2, 2]
97 -> [1, 5, 29, 135, 25, 6, 3, 4, 1, 1, 1, 1, 1, 2, 0, 0, 0, 0]
98 -> [1, 3, 20, 107, 6, 2, 2, 3, 2, 1, 1, 1, 1, 2, 1, 1, 1, 1]
99 -> [1, 5, 15, 75, 14, 4, 3, 3, 3, 3, 3, 3, 3, 5, 3, 3, 3, 3]
100 -> [1, 3, 20, 79, 19, 4, 4, 4, 3, 1, 1, 1, 1, 1, 1, 1, 1, 1]
```