# Lower bounds for explicit formulas

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# From the spectral side of Selberg's trace formula

The Maaß-Selberg relation gives the inner product formula

$$\int_{SL_2(\mathbf{Z})\backslash\mathbf{H}} \left|\Lambda^T E(z, \frac{1}{2} + it)\right|^2 dz = 2\log T - \frac{\varphi'}{\varphi}(\frac{1}{2} + it) + \mathrm{Im}(\varphi(\frac{1}{2} - it))\frac{T^{2it}}{t}$$

where $\varphi(s) = \frac{\Lambda(2s-1)}{\Lambda(2s)}$ and $\Lambda(s) = \pi^{-s/2}\Gamma(\frac{s}{2})\zeta(s)$, so

$$\frac{\varphi'}{\varphi}(\frac{1}{2} + it) = \frac{\Gamma'}{\Gamma}(it) - \frac{\Gamma'}{\Gamma}(\frac{1}{2} + it) + 2\frac{\zeta'}{\zeta}(2it) - 2\frac{\zeta'}{\zeta}(1 + 2it),$$

and with $T > 1$. We want to consider the integral

$$\mathcal{I}(T) := \frac{1}{4\pi}\int_{-\infty}^{\infty} h(t) \int_{SL_2(\mathbf{Z})\backslash\mathbf{H}} \left|\Lambda^T E(z, \frac{1}{2} + it)\right|^2 dz\ dt$$

and the game will be to find 'nice' $h(t)$.
**Assumption 1:** $h(t) \geq 0$ for all real $t$, so that $\mathcal{I}(T) \geq 0$.

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# From the spectral side of Selberg's trace formula

The integral $\mathcal{I}(T)$ breaks up into

$$\cdots = \frac{\log T}{2\pi}\int_{-\infty}^{\infty} h(t)dt \tag{1}$$

$$-\frac{1}{4\pi}\int_{-\infty}^{\infty}\left[\frac{\Gamma'}{\Gamma}(it) - \frac{\Gamma'}{\Gamma}(\frac{1}{2}+it) - 2\frac{\zeta'}{\zeta}(1+2it)\right]h(t)dt \tag{2}$$

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(2it)h(t)dt \tag{3}$$

$$+\frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{Im}(\varphi(\frac{1}{2}-it)h(t)\frac{T^{2it}}{t}dt \tag{4}$$

**Assumption 2:** $h(t)$ has sufficient decay as $t$ tends to $\pm\infty$.
Note the main dependence on $T$ is in (1); as (4) is roughly $o(1)$.

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\begin{aligned}
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& - \frac{1}{4\pi}\int_{-\infty}^{\infty}\Big[\frac{\Gamma'}{\Gamma}(it) - \frac{\Gamma'}{\Gamma}(\frac{1}{2}+it) - 2\frac{\zeta'}{\zeta}(1+2it)\Big] h(t)dt & (2)\\
& - \frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(2it)h(t)dt & (3)\\
& + \frac{1}{4\pi}\int_{-\infty}^{\infty}\mathrm{Im}(\varphi(\frac{1}{2}-it)h(t)\frac{T^{2it}}{t}dt & (4)
\end{aligned}
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# The sum over zeroes

Rewrite the term (3) as

$$-\frac{1}{2\pi}\int_{-\infty}^{\infty}\frac{\zeta'}{\zeta}(2it)h(t)dt = -\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}\frac{\zeta'}{\zeta}(2s)h(-is)ds$$

We want to apply the residue theorem to get

$$\sum_{\rho=\beta+i\gamma} h(\gamma - i\beta) - h(-\frac{i}{2}) - \frac{1}{2\pi i}\int_{\frac{1}{2}+\epsilon-i\infty}^{\frac{1}{2}+\epsilon+i\infty}\frac{\zeta'}{\zeta}(2s)h(-is)ds$$

where the sum $\rho$ runs over nontrivial zeroes of $\zeta(2s)$, and $\epsilon > 0$.
**Assumption** $2^+$**:** $h(t)$ is analytic with sufficient decay in a strip including $0 \leq \text{Im}(s) \leq \frac{1}{2} + \epsilon$. (As in Selberg.)

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# A qualitative lower bound

Putting this together, we have

$$\sum_{\rho} h(\gamma - i\beta) \geq -\frac{\log T}{2\pi} \int_{-\infty}^{\infty} h(t)dt + o(1) + C$$

where $C$ are the terms not depending on $T$, namely

$$C = h(-\frac{i}{2}) + \frac{1}{4\pi} \int_{-\infty}^{\infty} \Big[\frac{\Gamma'}{\Gamma}(it) - \frac{\Gamma'}{\Gamma}(\frac{1}{2} + it) - 2\frac{\zeta'}{\zeta}(1 + 2it)\Big] h(t)dt$$
$$+ \frac{1}{2\pi i} \int_{\frac{1}{2}+\epsilon-i\infty}^{\frac{1}{2}+\epsilon+i\infty} \frac{\zeta'}{\zeta}(2s)h(-is)ds,$$

and the expression converges for any fixed $T > 1$.

**Problem (Quantitative)**

*Find an explicit class test functions $h(t)$ for which the RHS is 0 for some $T > 1$.*

Note that since $h(t)$ is positive on $\mathbf{R}$, the integral $\int_{-\infty}^{\infty} h(t)dt$ is nonzero.

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# Final remarks

*Applications:* first zero of $L$-functions, lower bounds on discriminants, . . .

*Feature:* A priori independent of support. Yoshida, Bombieri: positvity for compactly supported functions on $(-\log 2/2, \log 2/2)$

*Generalizations:* In principle this is an Eisenstein series method. Roughly speaking,

- $SL_2$: (over number fields) Dedekind zeta functions.
- $GL_2$: Hecke $L$-functions.
- $GL_n$: Rankin-selberg $L$-functions.
- $G$: Langlands-Shahidi $L$-functions.

Error terms become more complicated depending on ramification.

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