Connectivity and distributions of three dimensional tilings

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Three Dimensional Tilings

Domino tilings of cubiculated Regions R:

- cubical complexes embedded as a finite polyhedron in $\mathbb{R}^N$
- connected oriented topological manifolds

Dimer covers of dual graph R^*

Q: Understand the space of tilings. What does a typical 3 dimensional tiling look like?

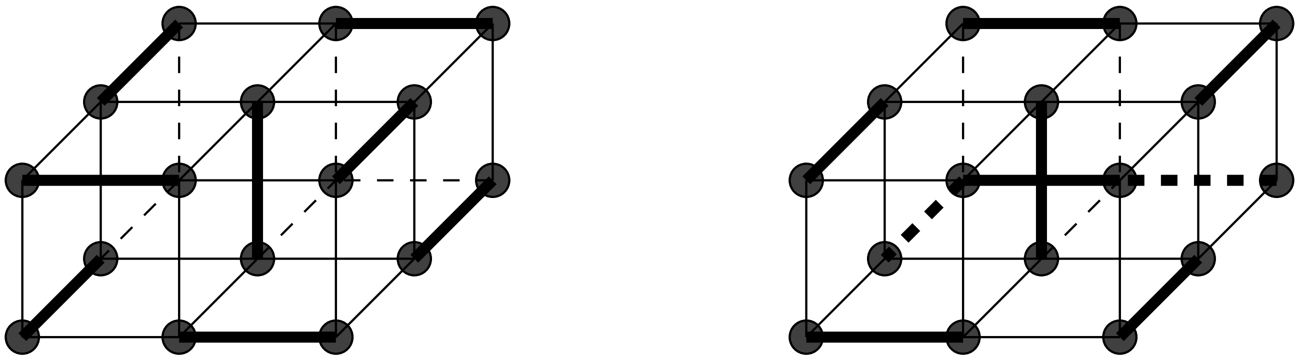
Focus: Connectivity by local moves. How and when can we move from one tiling to another?

Connectivity - Local Moves

Flip: Remove two adjacent parallel dominoes and place them back rotated within $2 \times 2 \times 1$ block.

Theorem: In two dimensions, any two tilings of a simply connected region are flip connected.

Not the case in 3d:



Two tilings of the $3 \times 3 \times 2$ box with no flips.

Connectivity - Local Moves

$3 \times 3 \times 2$ box:

Number of tilings: 229

Connected components: 3

Sizes: $227, 1, 1$

$4 \times 4 \times 4$ box:

Number of tilings: $5,051,532,105$

Connected components: 93

Sizes: $4,412,646,453$

$2 \times 310,185,960$

$2 \times 8,237,514$

$2 \times 718,308$

$2 \times 283,044$

$6 \times 2,576$

24×618

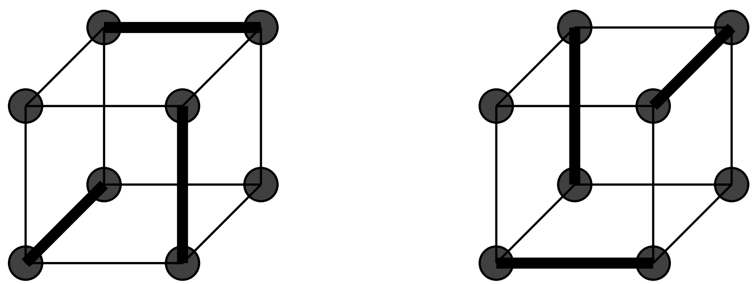
24×236

6×4

24×1

Connectivity - Local Moves

Trit: Remove and replace 3 dominoes, one parallel to each axis inside a $2 \times 2 \times 2$ box.



Question: Are tilings of three dimensional regions connected by flips and trits?

Connectivity - Local Moves

Question: Are all tilings of three dimensional regions connected by flips and trits?

In general, no.

Examples include tilings of:

- Cylinders: $\mathcal{D} \times [0, n]$
- Tori: $\mathbb{R}^3/\mathcal{L}$, $\mathcal{L} = 8\mathbb{Z}^3$.

Open:

- Boxes: $[0, L] \times [0, M] \times [0, N]$

Topological invariants

Two topological invariants: **Flux**, **Twist**.

Need notion of refinements.

Theorem (FKMS '16)

For two tilings t_0 and t_1 of R:

- There exists a sequence of flips and trits connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

Refinement

- R is refined by decomposing each cube into $5 \times 5 \times 5$ smaller cubes.
- t is refined by decomposing each domino into $5 \times 5 \times 5$ smaller dominoes, each parallel to the original.

Proposition

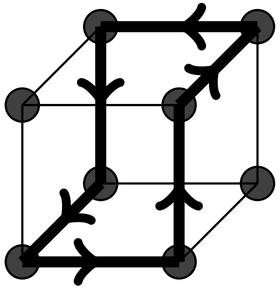
If t_0 and t_1 are connected by flips (resp. flips and trits) then their refinements are also connected by flips (resp. flips and trits).

(Converse false, examples in the $4 \times 4 \times 4$ box)

Flux - difference of tilings

For two tilings t_0, t_1

- $t_1 - t_0 :=$ union of tiles (with orientation of t_0 reversed).



Yields a system of cycles. (Ignore trivial 2-cycle.)

Homologically: $t_1 - t_0 \in Z_1(R^*; \mathbb{Z})$

Topological Invariant - Flux

Fix a base tiling $t_\oplus$.

$$\mathrm{Flux}(t) := [t - t_\oplus] \in H_1(R^*; \mathbb{Z})$$

flip $\rightsquigarrow$ boundary of a square

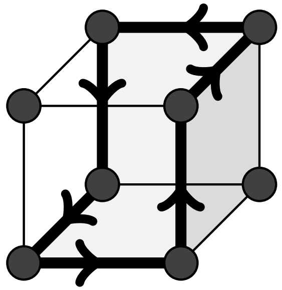
trit $\rightsquigarrow$ boundary of 3 squares

Proposition

If t_0 and t_1 differ by flips and trits then $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

Surface

A (discrete) Seifert surface for a pair t_0, t_1 is a connected embedded oriented topological surface S (mapped continuously and injectively into R^*) with boundary $t_0 - t_1$.



Proposition

t_0, t_1 if $\text{Flux}(t_0) = \text{Flux}(t_1)$ then there exists a discrete Seifert surface in some refinement of the pair.

flux through a surface

$$\varphi(v;t;S) = \mathrm{c}(v) \cdot \begin{cases} +1, & \text{end above } S \\ 0, & \text{end on } S \\ -1, & \text{end below } S \end{cases} \qquad \phi(t;S) = \sum_v \varphi(v;t;S)$$

$\mathrm{c}(v)$ is $+1$ if v is a black tile and -1 if v is a white tile.

Theorem

For S an embedded discrete surface with $\partial(S) = \emptyset$,

if $S = \partial(\text{manifold})$ then $\phi(t;S) = 0$.

Flux vs. flux

- $\phi(t; S)$ really only depends on the homology class of the surface.

Proposition

If $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ then $\phi(t_0; a) = \phi(t_1; a)$ for all $a \in H_2(R; \mathbb{Z})$.

Define the *modulus* of a tiling:

$$m := \mu(\mathrm{Flux}(t)) := \mathrm{gcd}_{a \in H_2} \phi(t; a)$$

(Twist is well-defined up to the modulus.)

Twist

Fix a base tiling $t_{\oplus}$:

$$\mathrm{Twist}(t) := \phi(t; t - t_{\oplus}) \ \in \mathbb{Z}/m\mathbb{Z}$$

Proposition

If $t_0 \rightsquigarrow \text{trit} \rightsquigarrow t_1$ then

$\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1) \pm 1$

- Intuitively, the twist records how "twisted" a tiling is by trits.
- If $\mathrm{Flux}(t) = 0$ then $\mathrm{Twist}(t) \in \mathbb{Z}$. (Boxes)

Main Theorem

Theorem

For two tilings t_0 and t_1 of R:

- There exists a sequence of flips and trits connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$.

- There exists a sequence of flips connecting refinements of t_0 and t_1 *if and only if* $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

- Proof: height functions, winding forms.

Questions

Q: How often are refinements necessary?

Conjecture

For $N \in 2\mathbb{Z}$, consider the cubical torus $R = \mathbb{Z}^3/(N \cdot \mathbb{Z}^3)$. Select two tilings t_0 and t_1 of R independently and uniformly at random.

- A: t_0 and t_1 are connected by flips;
- B: $\mathrm{Flux}(t_0) = \mathrm{Flux}(t_1)$ and $\mathrm{Twist}(t_0) = \mathrm{Twist}(t_1)$.

Then

$$\lim_{N\to\infty} \mathrm{Prob}[A|B] = 1.$$

Open: Are boxes flip and trit connected?

Stronger: Region inside a box?

Distribution of Twist

Q: How is the twist distributed?

- Normally distributed?
- Giant component?

Data of the $4 \times 4 \times 4$ box.