Colorful simplicial depth

Zuzana Patáková

joint work with Karim Adiprasito, Philip Brinkmann, Arnau Padrol, Pavel Paták, and Raman Sanyal

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1st December, Providence

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- $\varphi \colon \bigcup C_i \to \mathbb{R}^d$ $\quad\Rightarrow\quad$ $(\mathcal{C}, \varphi)$ is a colorful configuration

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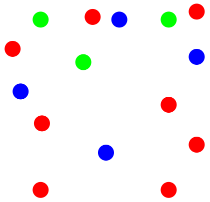

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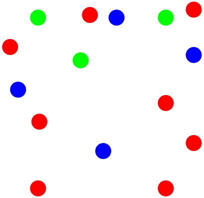

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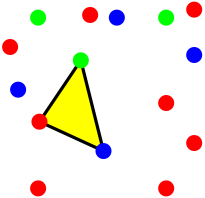

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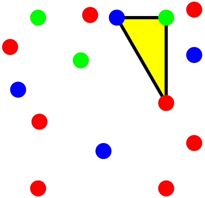

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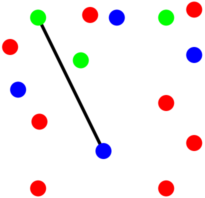

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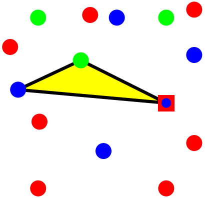

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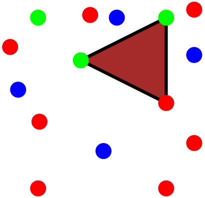

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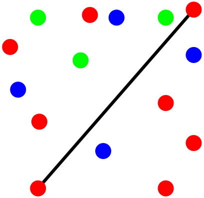

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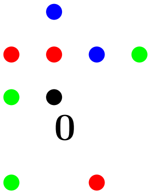

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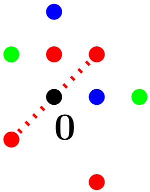

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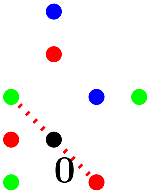

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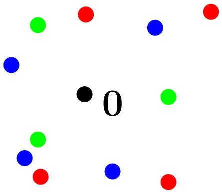

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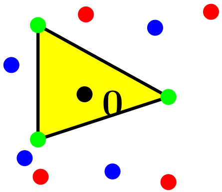

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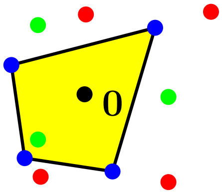

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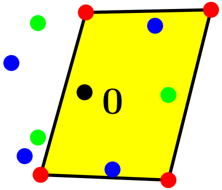

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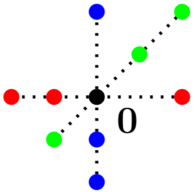

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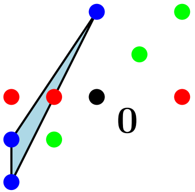

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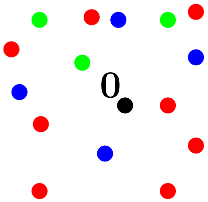

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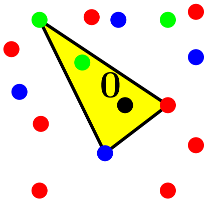

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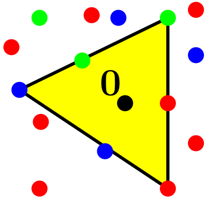

Another hitting simplex

- $\mathcal{C}$ is a colorful centered configuration
- colorful simplicial depth of $\mathcal{C}$ = cdepth($\mathcal{C}$)

  = **number of hitting simplices of** $\mathcal{C}$

Deza, Huang, Stephen and Terlaky '06

- Placing all C_i the same $\rightsquigarrow$ **simplicial depth** Liu '90
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Theorem (Colorful Carathéodory, Bárány '82)
There is always at least one hitting simplex. (cdepth ≥ 1)

Conjecture (Deza, Huang, Stephen, Terlaky, '06)
If Card C_0 = Card $C_1 = \ldots =$ Card $C_d = d+1$, *then*

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Deza et al: both bounds can be attained

Lower bound: Deza et al ['06], Bárány, Matoušek ['07], ...
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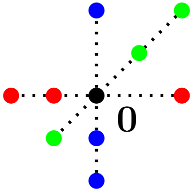

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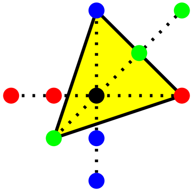

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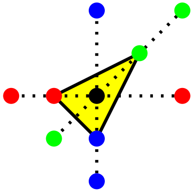

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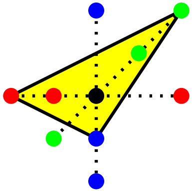

Topological reformulation

- $A =$ abstract simpl. complex of **all colorful** sets in $\bigcup C_i$
- $B =$ all sets $S \subset \bigcup C_i$ s.t. $\varphi(S)$ is **non-hitting**
- $f_i(K) =$ number of i-dim simplices in K
- $\text{cdepth}(\mathcal{C}) = f_d(A) - f_d(B)$
- $A^{(d-1)} = B^{(d-1)}$ $\Rightarrow$ for $i < d$ $\quad f_i(A) = f_i(B)$

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$$\Rightarrow f_d(A) = (-1)^d \left(\sum_{i=0}^{d} (-1)^i \widetilde{\beta}_i(A) + 1 - \sum_{i=0}^{d-1} (-1)^i f_i(A) \right)$$

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Our main Lemma: $\widetilde{\beta}_{d-1}(B) = 1$

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Lemma

$\widetilde{\beta}_{d-1}(B;\mathbb{Z}_2) = 1.$

Proof idea:

1. First show for a special configuration of points:

2. Use **flips** preserving $\widetilde{\beta}_{d-1}(B;\mathbb{Z}_2)$

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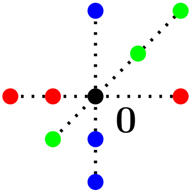

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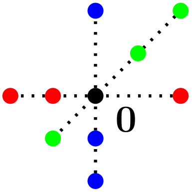

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Further connections

Further connections – normal surface theory

- normal d-fan = collection of polyhedral cones

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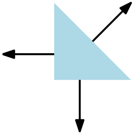

1-fan, given by normals of a triangle

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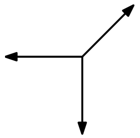

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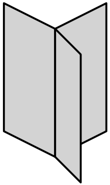

2-fan; halfplanes = leafs

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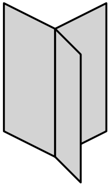
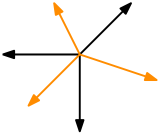
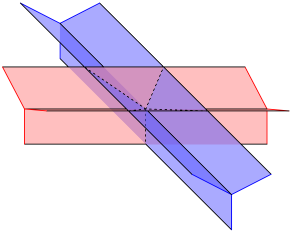


2-fan; halfplanes $=$ leafs

- two (and more) normal d-fans $\Rightarrow$ **common refinement**





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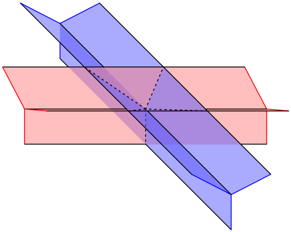

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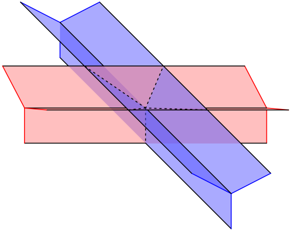

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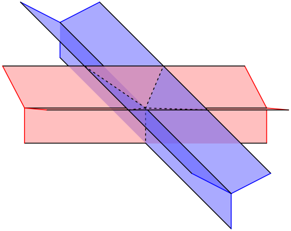

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- Minkowski sum
  $P_1 + P_2 + \ldots + P_k = \{p_1 + p_2 + \ldots + p_k \mid p_i \in P_i\} \subseteq \mathbb{R}^d$

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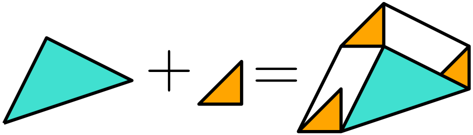

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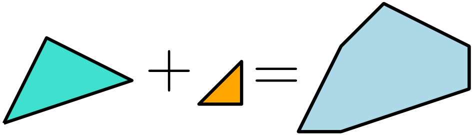

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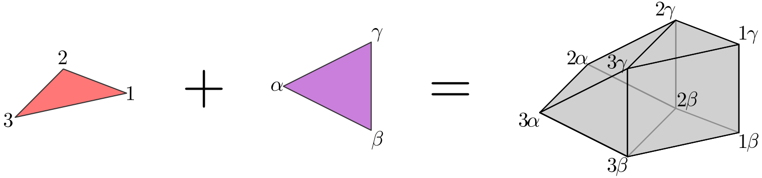

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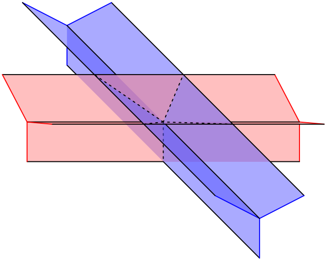

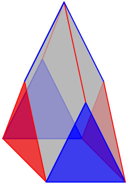

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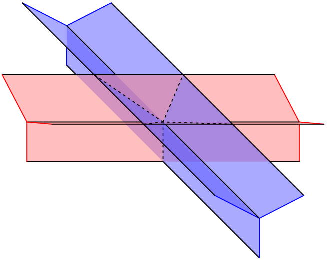
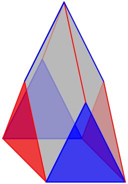
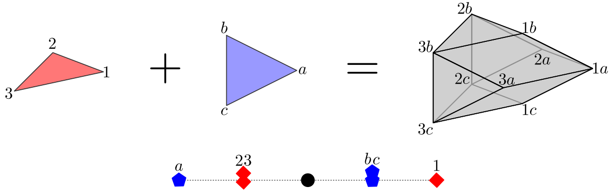




- **Reformulation:** number of rays = number of facets of Minkowski sum which correspond to a Minkow. sum of facets

- facets we are interested in

  = hitting simplices of the associated colorful Gale transform



- ⇒ Deza's bound $1 + \prod_{i=1}^{d-1}(|C_i| - 1)$ becomes $1 + 2^{d-1}$

  ⇒ **Burton's conjecture is true!!**

Proof idea

Proof of Main Lemma: Initial configuration

Lemma : $\widetilde{\beta}_{d-1}(B, \mathbb{Z}_2) = 1$

- Let $S \ni 0$ be a simplex with vertices $\mathbf{v}_0, \mathbf{v}_1, \ldots, \mathbf{v}_d$.
- $\varphi(C_i) = \{\mathbf{v}_i, -\mathbf{v}_i, -2\mathbf{v}_i, -3\mathbf{v}_i \ldots, -(|C_i| - 1)\mathbf{v}_i\}$.

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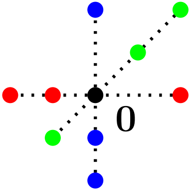

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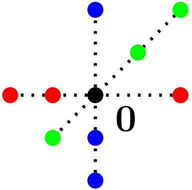

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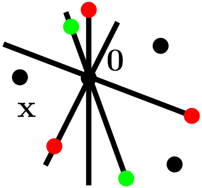

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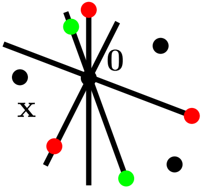

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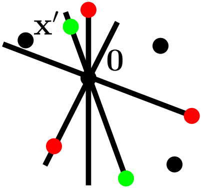

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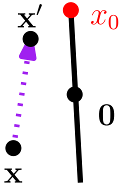

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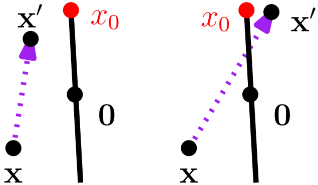

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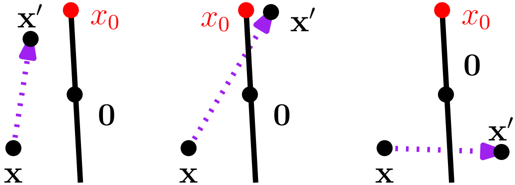
3. wild, otherwise



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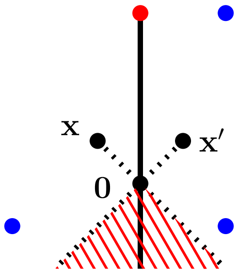

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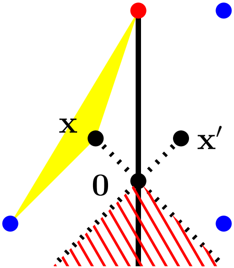

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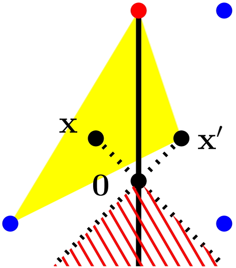

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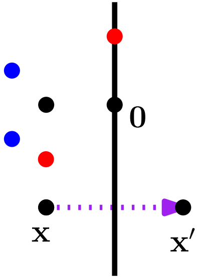

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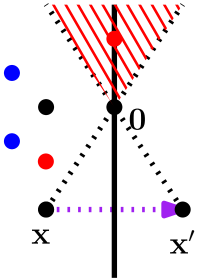

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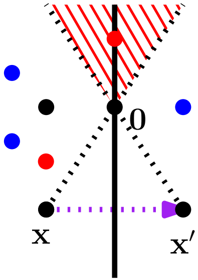

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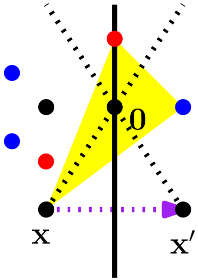

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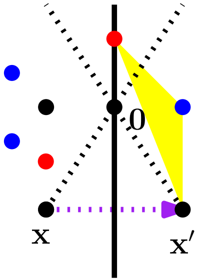

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Since $\widetilde{\beta}_{d-1}(B) = 1$, every $(d-1)$-cycle z in B can be expressed as

$$z = \sum_{i \in I} \partial \sigma_i + \sum_{j \in J} \partial \tau_j,$$

where $I \subseteq \{0, 1, \ldots, r\}$ and $J \subseteq \{1, \ldots, s\}$.

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$\partial\tau_i$ and $\partial\sigma_0$ boundaries in $B' \Rightarrow \partial\sigma_1, \ldots, \partial\sigma_r$ generate $\widetilde{H}_{d-1}(B')$.

Clearly $\partial\sigma_1$ is not zero homologous, therefore $\widetilde{\beta}_{d-1}(B') \geq 1$.

Lemma: For every $k > 0$, the cycle $\partial\sigma_1 + \partial\sigma_k$ is contained in a subcomplex C with $\widetilde{\beta}_{d-1}(C) = 0$.

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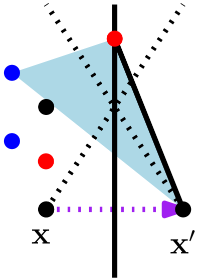

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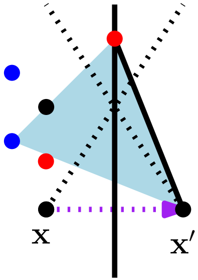

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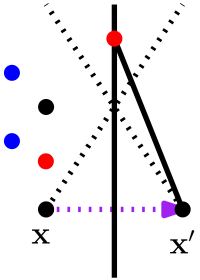

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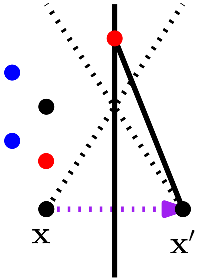

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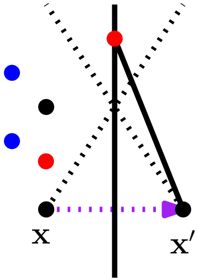

$\Rightarrow$ all $(d-1)$-cycles in C are zero homologous
$\Rightarrow$ $\partial\sigma_1$ and $\partial\sigma_k$ are homologous in B' for all k and $\widetilde{\beta}_{d-1}(B') = 1$ as claimed. □

Thank you for your attention!