Prismatic Maps for the Topological Tverberg Conjecture

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Joint work with Uli Wagner
Geometrically the notion of homotopy is a horrible idea, because during a homotopy a nice embedding gets all mangled up. But the virtue of homotopy theory is that the homotopy classes of maps are often finite or finitely generated, and frequently computable, and so out of the mess we get something interesting.
Seminar on Combinatorial Topology
by E.C. ZEEMAN

Chapter 8: EMBEDDING AND UNKNOTTING

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Hope

Geometry ≅ Algebra
General Problem:

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\[ p = f(x_1) = \cdots = f(x_r) \]

A map $f : K \to \mathbb{R}^d$ without $r$-fold intersection is called an $r$-embedding.
Example: $f : K^2 \to \mathbb{R}^3$
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\[ \mathbb{R}P^2 \xrightarrow{f} \] Boy’s Surface
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$K = \text{real projective plane } \mathbb{RP}^2$

$\mathbb{RP}^2 \xrightarrow{f} $ 2-fold intersection

Boy's Surface
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\( \mathbb{R}P^2 \xrightarrow{f} \)  

(2-fold intersection) 
(3-fold intersection) 
(Unique) 

Boy’s Surface
Example: \( f: K^2 \rightarrow \mathbb{R}^3 \)

\( K = \) real projective plane \( \mathbb{RP}^2 \)

\( f: \mathbb{RP}^2 \rightarrow \mathbb{R}^3 \) is a 4-embedding (no 4-fold intersections)
Classical Case: Maps without 2-fold intersections
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Goal: Find $f : K \to \mathbb{R}^d$ continuous & injective (i.e., $f$ is an embedding)
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Theorem (van Kampen–Shapiro–Wu):

\[ \exists f: K^m \hookrightarrow \mathbb{R}^{2m} \iff \exists \tilde{f}: K_\delta \times 2 \to S_2 S^{2m-1} \]

provided $m \neq 2$. 
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‘easy’ **Proposition** The existence of \( K_\delta \times 2 \to \mathbb{S}_2 S^{2m-1} \) is algorithmically solvable.

**Corollary.** The existence of an embedding \( K^m \hookrightarrow \mathbb{R}^{2m} \) is algorithmically solvable, provided \( m \neq 2 \).
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**Goal:** Find $f : K \rightarrow \mathbb{R}^d$ continuous & without $r$-fold intersection (i.e., $f$ is an $r$-embedding)

An **necessary condition** for the existence of $f$:

1) Define the $r$-fold deleted product of $K$:

$$K_{\delta}^r := \{ \sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \in K \text{ and } \sigma_i \cap \sigma_j = \emptyset \} \subset K^r$$

$$f(K) \subset \mathbb{R}^3$$
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2) Given an $r$-embedding $f: K \rightarrow \mathbb{R}^d$, define

$$ \tilde{f}: K^\times_r \rightarrow \mathbb{R}^{d \times r} $$

$$(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$$

$$ f(K) \subset \mathbb{R}^3 $$
Two properties of $\tilde{f}$

$\tilde{f} : K^\times_{d} \rightarrow \mathbb{R}^{d \times r}$

$(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$
Two properties of $\tilde{f}$

$$\tilde{f} : \quad K_δ^{×r} \rightarrow \mathbb{R}^{d×r}$$

$$(x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$$

A) The symmetric group $\mathfrak{S}_r$ acts on both $K_δ^{×r}$ and $\mathbb{R}^{d×r}$ by permutation of the coordinates.

$\tilde{f}$ is compatible with both actions (i.e., $\tilde{f}$ is $\mathfrak{S}_r$-equivariant):

For all $\rho \in \mathfrak{S}_r$

$$\tilde{f} \circ \rho = \rho \circ \tilde{f}$$
Two properties of $\tilde{f}$

$\tilde{f}: K_\delta^\times r \rightarrow \mathbb{R}^{d\times r}$

$\quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad \quad (x_1, \ldots, x_r) \mapsto (fx_1, \ldots, fx_r)$

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B) $(x_i \in \sigma_i \in K$ and $\sigma_i \cap \sigma_j = \emptyset) \Rightarrow$ all the $x_i$ are distinct

$f$ is an $r$-embedding $\Rightarrow \neg (fx_1 = \cdots = fx_r)$
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Hence:

\[ \tilde{f}: \quad K_δ^{\times r} \rightarrow \mathfrak{S}_r \mathbb{R}^{d \times r} \setminus \{(x, \ldots, x) \mid x \in \mathbb{R}^d\} \]
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B) $(x_i \in \sigma_i \in K \text{ and } \sigma_i \cap \sigma_j = \emptyset) \Rightarrow$ all the $x_i$ are distinct

$f$ is an $r$-embedding $\Rightarrow \lnot(f x_1 = \cdots = f x_r)$

Hence:

\[ \tilde{f} : \quad K_\delta^{\times r} \rightarrow \mathfrak{S}_r \quad \mathbb{R}^{d \times r} \setminus \{(x, \ldots, x) \mid x \in \mathbb{R}^d\} \cong S^{(r-1)d-1} \]
$f : K^m \to \mathbb{R}^d$ such that for all $\sigma_1, \ldots, \sigma_r \in K$ with $\sigma_i \cap \sigma_j = \emptyset$

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\[\Downarrow\]

\[\exists \tilde{f} : K_\delta^r \to \mathcal{S}_r S^{(r-1)d-1}\]
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provided $m = (r - 1)k$, $d = rk$ and $k \geq 3$.
$f$ is an almost $r$-embedding

$f : K^m \to \mathbb{R}^d$ such that for all $\sigma_1, \ldots, \sigma_r \in K$ with $\sigma_i \cap \sigma_j = \emptyset$

$f_{\sigma_1} \cap \cdots \cap f_{\sigma_r} = \emptyset$

provided $m = (r - 1)k$, $d = rk$ and $k \geq 3$
Theorem:

\[ \exists f : K^{(r-1)k} \to \mathbb{R}^{rk} \text{ almost } r\text{-embedding} \]
\[ \iff \]
\[ \exists \tilde{f} : K^{\times r} \delta \to \mathcal{S}_r S^{(r-1)rk-1} \]

provided \( k \geq 3 \).
Theorem:

\[ \exists f : K^{(r-1)k} \rightarrow \mathbb{R}^{rk} \text{ almost } r\text{-embedding} \]

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provided \( k \geq 3 \).

geometric problem (map without intersection) \( \iff \) algebraic problem (equivariant map)
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Corollary. The existence of \( f : K^{(r-1)k} \rightarrow \mathbb{R}^{rk} \text{ almost } r\text{-embedding} \) is algorithmically solvable, provided \( k \geq 3 \).
Our Main Tool:
an $r$-fold analogue of the Whitney Trick
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an \( r \)-fold analogue of the Whitney Trick

Classical Whitney Trick with two balls \( \sigma^p \) and \( \tau^q \) in \( \mathbb{R}^{p+q} \):

\[ p, q \geq 3 \]
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Classical Whitney Trick with two balls $\sigma^p$ and $\tau^q$ in $\mathbb{R}^{p+q}$:

push $\sigma^p$ along the Whitney Disk $D^2$
What happens with more than two balls?
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Whitney trick for two balls

Then, use that $\sigma^6$ is “flat” (codimension $\geq 3$) to extend the solution to $\mathbb{R}^9$. 
What happens with more than two balls?

Problem: $\sigma \cap \tau$ and $\sigma \cap \mu$ are, in general, not connected spaces

Whitney trick for two balls

Then, use that $\sigma^6$ is “flat” (codimension $\geq 3$) to extend the solution to $\mathbb{R}^9$. 
Piping + Unpiping Trick
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$\tau^p$

$\mathbb{R}^{p+3}$

1-handle
Piping + Unpiping Trick

$\mathbb{R}^{p+3}$
Piping + Unpiping Trick

\[ \tau^p + \text{the two handles} \simeq \text{a } p\text{-ball} \]
Back to the intersection problem
Back to the intersection problem
Back to the intersection problem

1-handle

complementary 2-handle
Back to the intersection problem

Hence, we can add 1-handles on $\sigma \cap \tau$. I.e., we can make $\sigma \cap \tau$ connected.
We can assume $\sigma \cap \tau$ and $\sigma \cap \mu$ are connected. Hence we can use the classical Whitney trick to solve the 3-balls situation, i.e., to remove triple intersection points.
**r-fold Whitney Trick**

Given $r$ balls $B_1, \cdots, B_r$ mapped by a $f$ into $\mathbb{R}^d$ in general position

$$f : B_1 \sqcup \cdots \sqcup B_r \to \mathbb{R}^d$$

with

$$d - \dim(B_i) \geq 3 \quad \text{and} \quad \sum_i d - \dim(B_i) = d.$$

If

$$f(B_1) \cap \cdots \cap f(B_r) = \{x, y\}$$

two points of opposite signs. Then we can remove these two points by a move along a 2-dimensional cone ($\approx$ “Whitney disk”).

In particular, we can avoid any codimension $\geq 3$ object in $\mathbb{R}^d$ during this move.
classical Whitney Trick $\Rightarrow$ first part of Van Kampen
Embeddability ($k \neq 2$):

$$K^k \rightarrow \mathbb{R}^{2k}$$ almost $2$-embeds

$$\iff$$

$$K_{\delta}^{\times 2} \rightarrow \mathcal{S}_2 S^{2k-1}$$
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Embeddability \((k \neq 2)\):

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\[ \Leftrightarrow \]

\[ K_\delta^\times 2 \rightarrow \mathcal{S}_2 \ S^{2k-1} \]

r-fold Whitney Trick \Rightarrow\ For \( r, k \geq 3 \),

\[ K^{(r-1)k} \rightarrow \mathbb{R}^{rk} \text{ almost } r\text{-embeds} \]

\[ \Leftrightarrow \]

\[ K_\delta^\times r \rightarrow \mathcal{S}_r \ S^{r(r-1)k-1} \]
classical Whitney Trick $\Rightarrow$ first part of Van Kampen Embeddability ($k \neq 2$):

$$K^k \to \mathbb{R}^{2k} \text{ almost } 2\text{-embeds} \iff K \times 2 \delta \to S^2 S^{2k-1}$$

r-fold Whitney Trick $\Rightarrow$ For $r, k \geq 3$,

$$K^{(r-1)k} \to \mathbb{R}^{rk} \text{ almost } r\text{-embeds} \iff K^r \to S^{r(r-1)k-1} \text{ check a system of linear equations over } \mathbb{Z}$$
Application: Topological Tverberg

Topological Tverberg Conjecture: Given \( r, d \geq 2 \), there exists no almost \( r \)-embedding
\[
\Delta^{(r-1)(d+1)} \rightarrow \mathbb{R}^d.
\]

Example for \( r = 2 \)

\( \Delta^3 \)

\[ f \sigma_1 \cap f \sigma_2 \neq \emptyset \]

Hence \( f \) is not an almost 2-embedding
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The conjecture holds for $r = \text{prime}^{\text{power}}$ (Ozaydin87)
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$$\Delta^{(r-1)(d+1)} \to \mathbb{R}^d.$$ 

(Ozaydin 1987) If $r$ is not a prime power and $\dim X \leq d(r-1)$ with free $\mathfrak{S}_r$-action, then $X \to_{\mathfrak{S}_r} S^{(r-1)d-1}$
**Application:** Topological Tverberg

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(M-Wagner) Provided $k \geq 3$:

$$K^\times_r \to S^{(r-1)rk-1} \Leftrightarrow K^{(r-1)k} \to \mathbb{R}^{rk}$$ almost $r$-embeds
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(Gromov 2010, Blagojevic-Frick-Ziegler 2014)

$$\Delta^{(3r+2)(r-1)} \leq 3(r-1) \rightarrow \mathbb{R}^{3r} \text{ almost } r\text{-embeds}$$

$$\Rightarrow \Delta^{(3r+2)(r-1)} \rightarrow \mathbb{R}^{3r+1} \text{ almost } r\text{-embeds}.$$
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\Delta^{(r-1)(d+1)} \rightarrow \mathbb{R}^d.
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(Ozaydin 1987) If \( r \) is not a prime power and \( \text{dim } X \leq d(r-1) \) with free \( S_r \)-action, then \( X \rightarrow S_r S^{(r-1)d-1} \)

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K_{\delta} \times_r \rightarrow S^{(r-1)rk-1} \iff K^{(r-1)k} \rightarrow \mathbb{R}^{rk} \text{ almost } r\text{-embeds}
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Frick’s observation:

\((O) + (MW) + (G) \Rightarrow \) there exists an almost 6-embedding \( \Delta^{100} \rightarrow \mathbb{R}^{19} \)
Application: Topological Tverberg

Topological Tverberg Conjecture: Given \( r, d \geq 2 \), there exists no almost \( r \)-embedding
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\Delta^{(r-1)(d+1)} \to \mathbb{R}^d.
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(Ozaydin 1987) \( (\Delta^{100}_{\leq 15})_\delta \times 6 \to \mathcal{G}_6 \mathcal{S}^{89} \)

(M-Wagner) Provided \( k \geq 3 \):
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K^\times_r \to \mathcal{G}_r \mathcal{S}^{(r-1)rk-1} \iff K^{(r-1)k} \to \mathbb{R}^{rk} \text{ almost } r\text{-embeds}
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\Delta^{(3r+2)(r-1)} \leq 3(r-1) \to \mathbb{R}^{3r} \text{ almost } r\text{-embeds}
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Topological Tverberg Conjecture: Given $r, d \geq 2$, there exists no almost $r$-embedding

$$\Delta^{(r-1)(d+1)} \to \mathbb{R}^d.$$ 

(Ozaydin 1987) $(\Delta_{\leq 15}^{100})^{\times 6} \to S^89$

(M-Wagner)

$(\Delta_{\leq 15}^{100})^{\times 6} \to S^89 \Rightarrow \Delta_{\leq 15}^{100} \to \mathbb{R}^{18}$ almost 6-embeds

(Gromov 2010, Blagojevic-Frick-Ziegler 2014)

$\Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r}$ almost $r$-embeds

$\Rightarrow \Delta^{(3r+2)(r-1)} \to \mathbb{R}^{3r+1}$ almost $r$-embeds.

Frick’s observation:

(O) + (MW) + (G) $\Rightarrow$ there exists an almost 6-embedding $\Delta^{100} \to \mathbb{R}^{19}$
Application: Topological Tverberg

Topological Tverberg Conjecture: Given $r, d \geq 2$, there exists no almost $r$-embedding

$$\Delta^{(r-1)(d+1)} \to \mathbb{R}^d.$$ 

(Ozaydin 1987) \((\Delta_{\leq 15}^{100}) \times 6 \to \mathcal{S}_6 S^{89}\)

(M-Wagner) \((\Delta_{\leq 15}^{100})_\delta \times 6 \to \mathcal{S}_6 S^{89} \Rightarrow \Delta_{\leq 15}^{100} \to \mathbb{R}^{18} \text{ almost 6-embeds}\)

(Gromov 2010, Blagojevic-Frick-Ziegler 2014) \(\Delta_{\leq 15}^{100} \to \mathbb{R}^{18} \text{ almost 6-embeds} \Rightarrow \Delta^{100} \to \mathbb{R}^{19} \text{ almost 6-embeds.}\)

Frick’s observation: \((O) + (MW) + (G) \Rightarrow \text{ there exists an almost 6-embedding } \Delta^{100} \to \mathbb{R}^{19}\)
Theorem (Avvakumov-M-Skopenkov-Wagner). For $d \geq 2r$ and $r$ not a prime power, there exists an almost $r$-embedding
\[ \Delta^{(d+1)(r-1)} \to \mathbb{R}^d \]
Theorem (Avvakumov-M-Skopenkov-Wagner). For $d \geq 2r$ and $r$ not a prime power, there exists an almost $r$-embedding

$$\Delta^{(d+1)(r-1)} \to \mathbb{R}^d$$

Minimal counterexample: almost 6-embedding $\Delta^{65} \to \mathbb{R}^{12}$. 
**Theorem** (Avvakumov-M-Skopenkov-Wagner). For $d \geq 2r$ and $r$ not a prime power, there exists an almost $r$-embedding

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**Minimal counterexample**: almost 6-embedding $\Delta^{65} \rightarrow \mathbb{R}^{12}$.

What happens for $d \leq 11$?
**Theorem** (Avvakumov-M-Skopenkov-Wagner). For $d \geq 2r$ and $r$ not a prime power, there exists an almost $r$-embedding

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**Minimal counterexample**: almost 6-embedding $\Delta^{65} \to \mathbb{R}^{12}$.

What happens for $d \leq 11$?

First open case of the conjecture: almost 6-embedding $\Delta^{15} \to \mathbb{R}^2$. I.e., a drawing of $K_{16}$ without
How the minimal counterexample $\Delta^{65} \to \mathbb{R}^{12}$ was obtained?

Two new tools:
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1) *Prismatic maps* (forcing codimension)

All the Tverberg partitions are made of triangles
How the minimal counterexample $\Delta^{65} \rightarrow \mathbb{R}^{12}$ was obtained?

Two new tools:

1) Prismatic maps (forcing codimension)

All the Tverberg partitions are made of triangles

2) A codimension 2 (!) Whitney Trick

(Avvakumov-M-Skopenkov-Wagner) Provided $k \geq 2$ and $r \geq 3$:

$\exists K^{(r-1)k} \rightarrow \mathbb{R}^{rk}$ almost $r$-embedding $\Leftrightarrow K_{r}^{r-1} \rightarrow S^{(r-1)rk-1}$
Frick’s counterexample $\Delta^{100} \rightarrow \mathbb{R}^{19}$

$19 = 6 \cdot 3 + 1$
Frick's counterexample $\Delta^{100} \rightarrow \mathbb{R}^{19}$

$19 = 6 \cdot 3 + 1$

- Ozaydin (smallest non-prime power)
- M-Wagner r-fold Whitney Trick
- Gromov trick, Constraint method (BFZ)
M-Wagner prismatic counterexample $\Delta^{95} \rightarrow \mathbb{R}^{18}$

$18 = 6 \cdot 3 + 0$

Ozaydin (smallest non-prime power)

M-Wagner r-fold Whitney Trick

prismatic maps
Avvakumov-M-Skopenkov-Wagner prismatic codim 2 counterexample $\Delta^{65} \rightarrow \mathbb{R}^{12}$

$12 = 6 \cdot 2 + 0$

Ozaydin (smallest non-prime power)

codim 2 r-fold Whitney Trick

prismatic maps
Avvakumov-M-Skopenkov-Wagner prismatic codim 2 counterexample $\Delta^{65} \to \mathbb{R}^{12}$

$12 = 6 \cdot 2 + 0$

What happens in lower dimension ($2 \leq d \leq 11$) remains a mystery...
In another direction...
In another direction...

The van-Kampen-Shaprio-Wu theorem was vastly extended in the 60s
In another direction...

The van-Kampen-Shaprio-Wu theorem was vastly extended in the 60s.

**Theorem (Haefliger-Weber 60’s):**

\[ \exists f : K^m \hookrightarrow \mathbb{R}^d \iff \exists \tilde{f} : K_\delta \times^2 \to S_2 S^{d-1} \]

provided \( 2d \geq 3m + 3 \) (\( = \) metastable range)
In another direction...

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provided \(2d \geq 3m + 3\) (＝metastable range)

**Theorem** (Cadek-Krcal-Vokrinek 13) The existence of \(\delta \times K^2 \rightarrow \mathbb{S}_2 \mathbb{S}^{d-1}\) is algorithmically solvable.
In another direction...

The van-Kampen-Shaprio-Wu theorem was vastly extended in the 60s

**Theorem** (Haefliger-Weber 60’s):

\[ \exists f : K^m \hookrightarrow \mathbb{R}^d \iff \exists \tilde{f} : K_\delta \times 2 \rightarrow \mathbb{S}_2 \mathbb{S}^{d-1} \]

provided \(2d \geq 3m + 3\) (=metastable range)

**Theorem** (Cadek-Krčal-Vokrinek 13) The existence of \(K_\delta \times 2 \rightarrow \mathbb{S}_2 \mathbb{S}^{d-1}\) is algorithmically solvable.

**Corollary.** The existence of an embedding \(K^m \hookrightarrow \mathbb{R}^d\) is algorithmically solvable, provided \(d \geq 1.5m\)
Theorem (M-Wagner)

\[ \exists f : K^m \rightarrow \mathbb{R}^d \text{ almost } r\text{-embedding} \iff \exists \tilde{f} : K_\delta^r \rightarrow \mathcal{S}_r S(r-1)d-1 \]

provided \( rd \geq (r+1)m + 3 \) (\( = \) \( r \)-metastable range).
\textbf{Theorem (M-Wagner)}

\[ \exists f : K^m \rightarrow \mathbb{R}^d \text{ almost } r\text{-embedding} \iff \exists \tilde{f} : K_{\delta}^\times r \rightarrow \mathcal{G}_r S^{(r-1)d-1} \]

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\textbf{Theorem (Filakovksy-Vokrinek).} The existence of
\[ K_{\delta}^\times r \rightarrow \mathcal{G}_r S^{(r-1)d-1} \]

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Theorem (M-Wagner)

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Corollary. The existence of \( f : K^m \to \mathbb{R}^d \text{ almost } r\text{-embedding} \)

is algorithmically solvable, provided \( d \gtrsim \frac{r+1}{r} m \).
Theorem (M-Wagner)\n
\[ \exists f : K^m \to \mathbb{R}^d \text{ almost } r\text{-embedding} \iff \exists \tilde{f} : K^\times_r \to \mathcal{S}_r S(r-1)d-1 \]

provided \( rd \geq (r + 1)m + 3 \) (\( = \) r-metastable range).

Theorem (Filakovksy-Vokrinek). The existence of \( K^\times_r \to \mathcal{S}_r S(r-1)d-1 \) is \underline{algorithmically solvable}.

Corollary. The existence of \( f : K^m \to \mathbb{R}^d \text{ almost } r\text{-embedding} \)

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THANK YOU!!