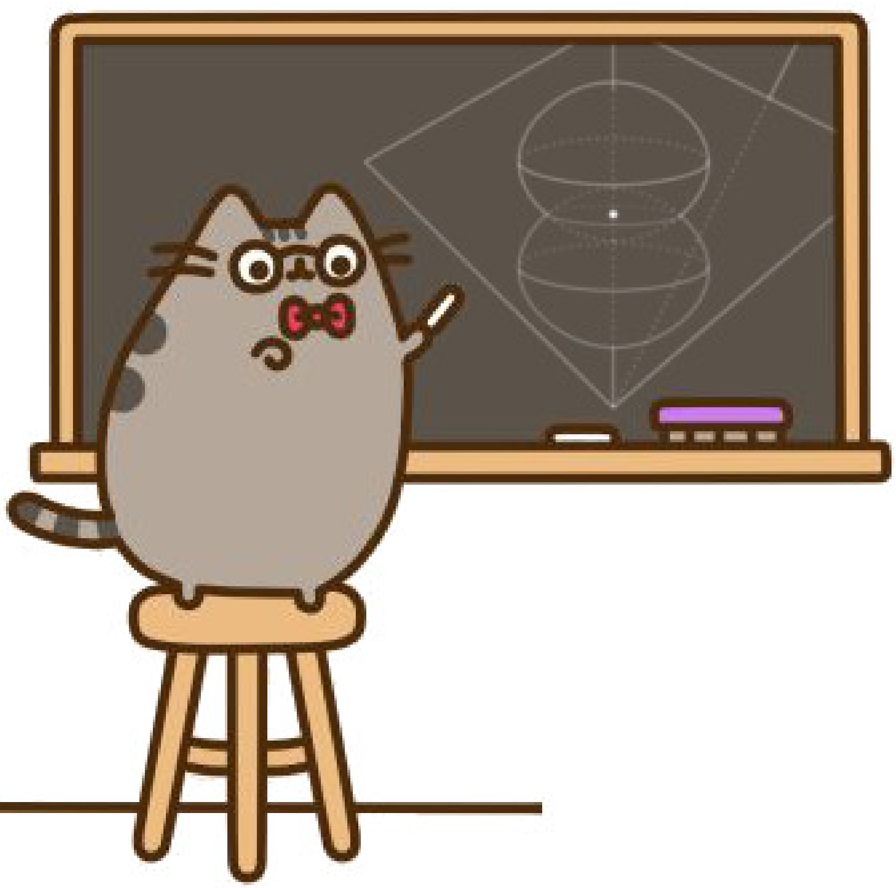
Prismatic Maps for the Topological Tverberg Conjecture



Isaac Mabillard
Joint work with Uli Wagner

Seminar on Combinatorial Topology

by E.C. ZEEMAN

Chapter 8 : EMBEDDING AND UNKNOTTING

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Geometry $\to$ Algebra

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Hope

Geometry $\cong$ Algebra

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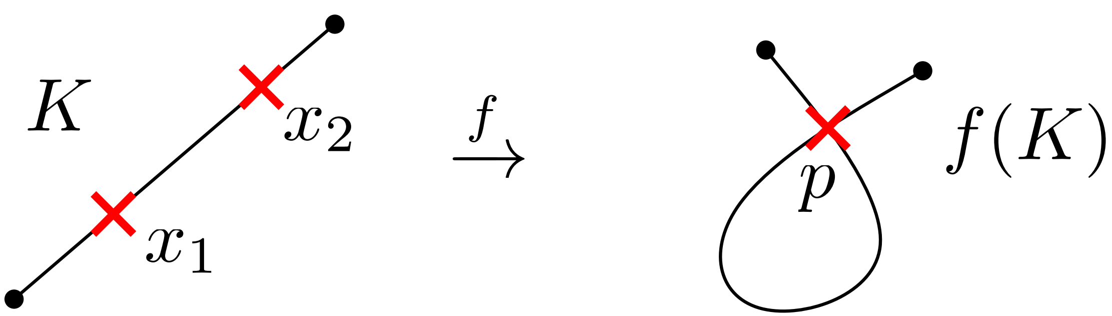

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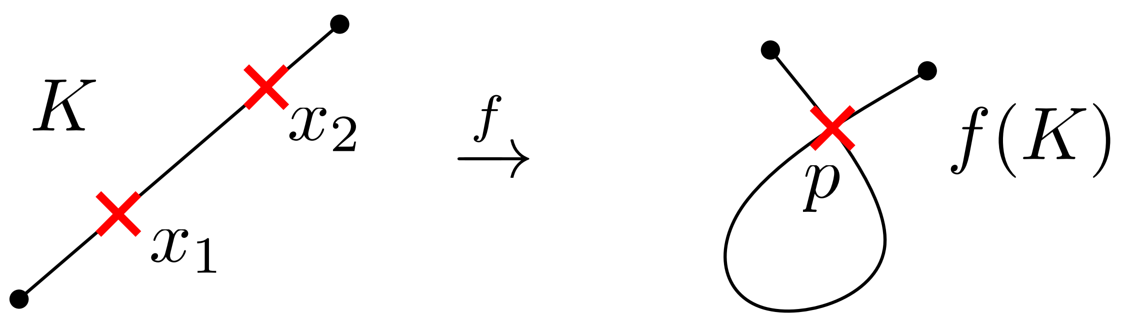

A map $f : K \to \mathbb{R}^d$ without r-fold intersection is called r-embedding

Example: $f\colon K^2 \to \mathbb{R}^3$

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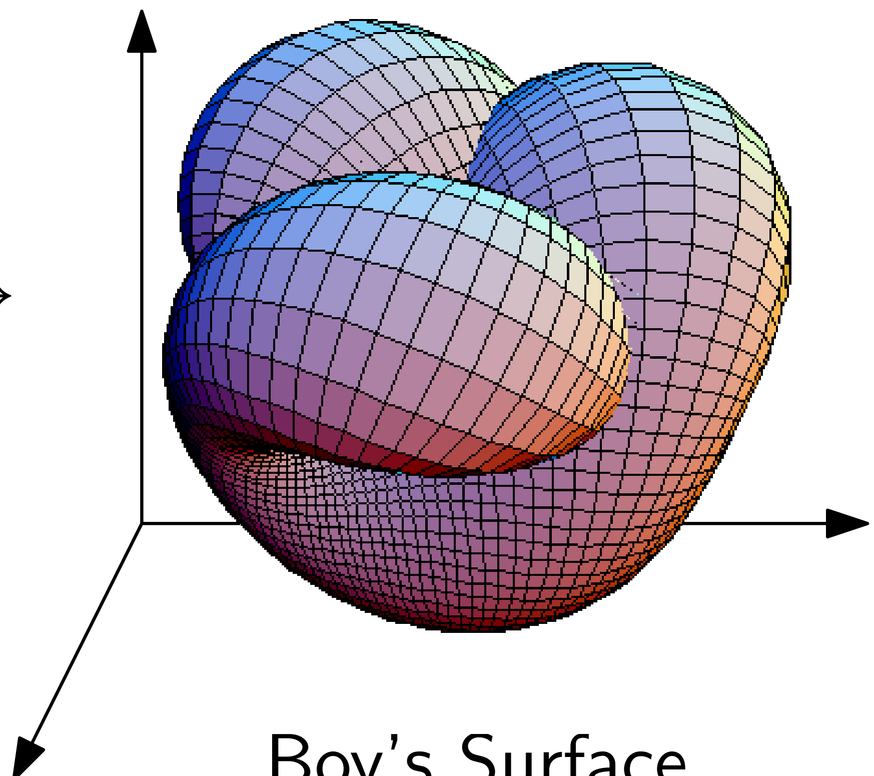

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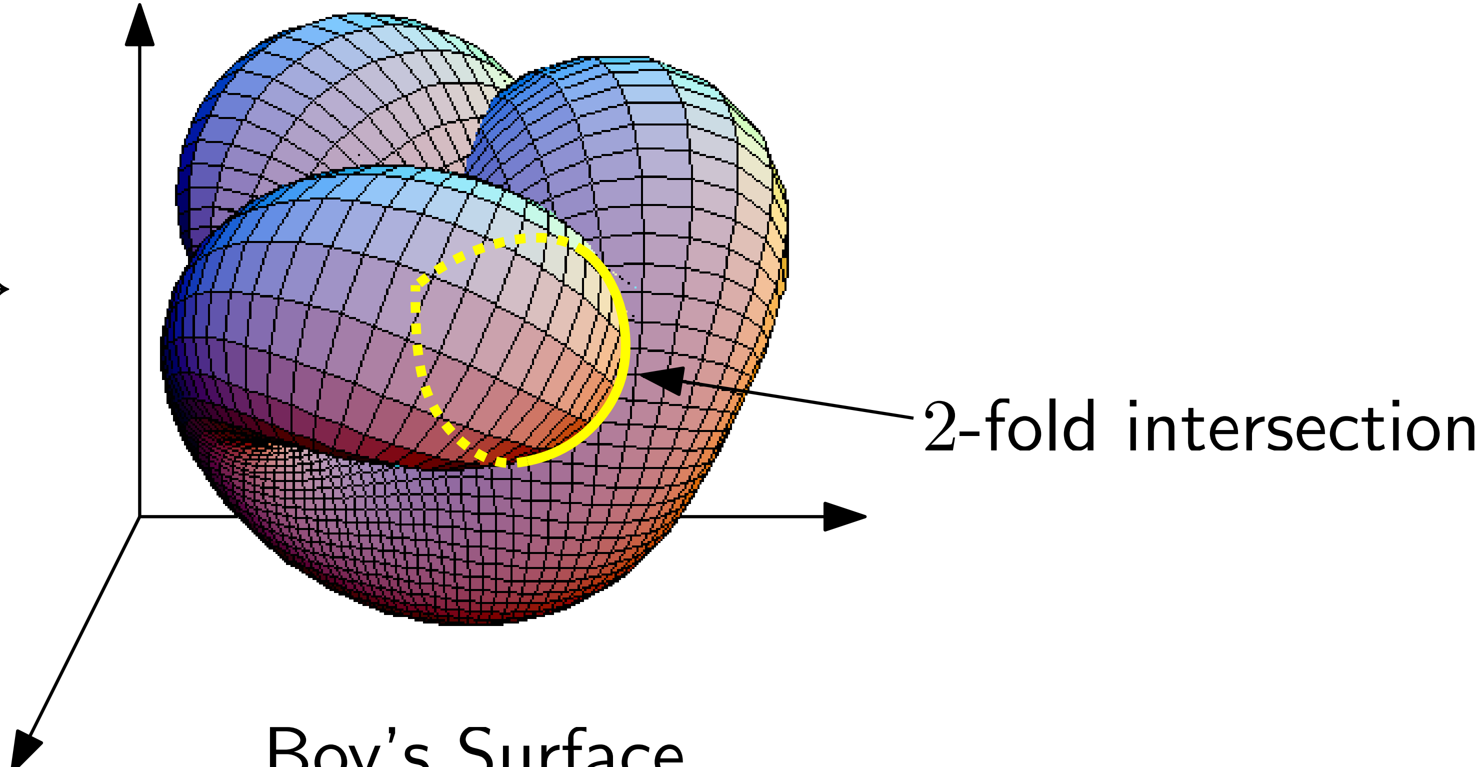

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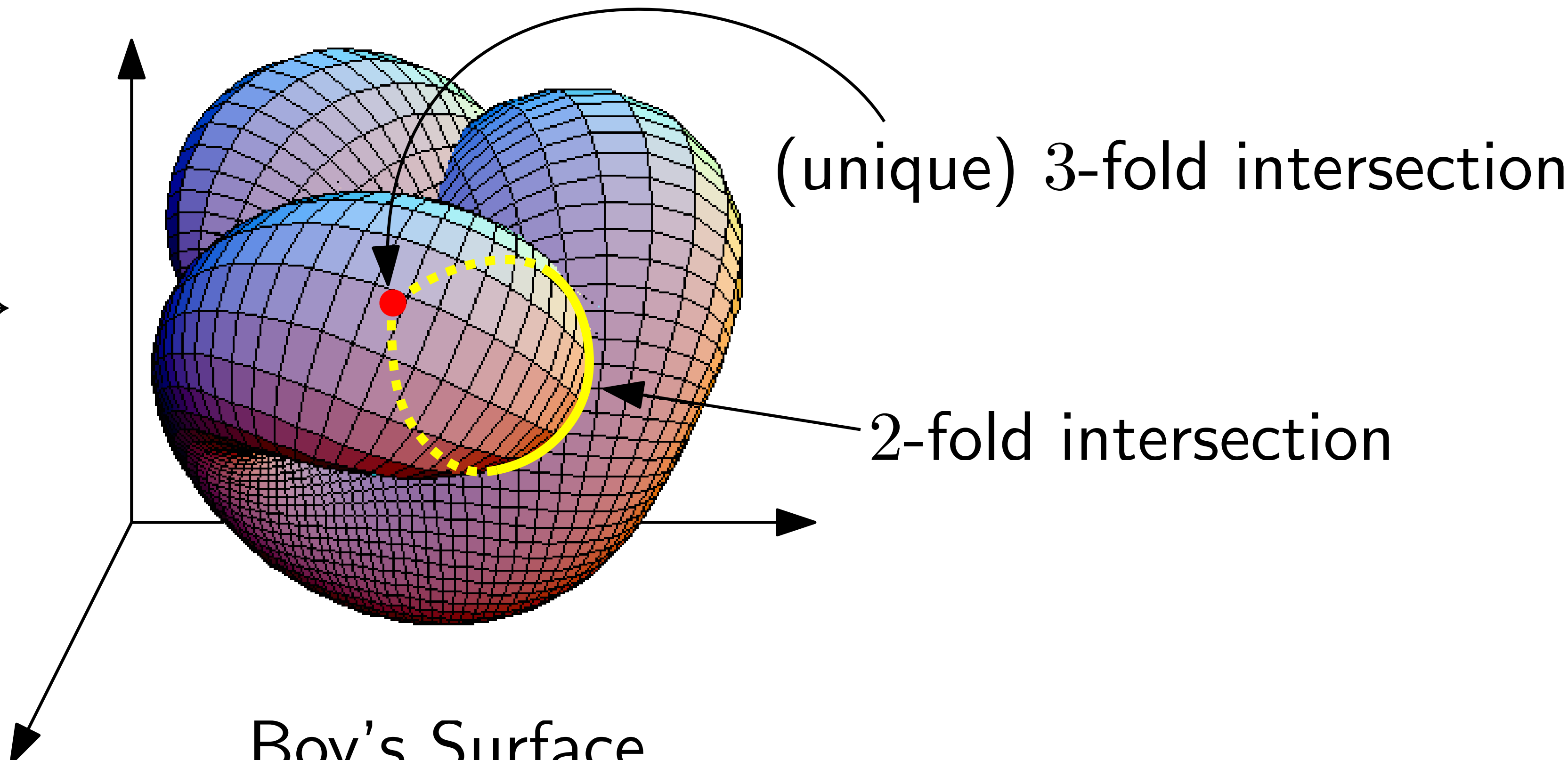

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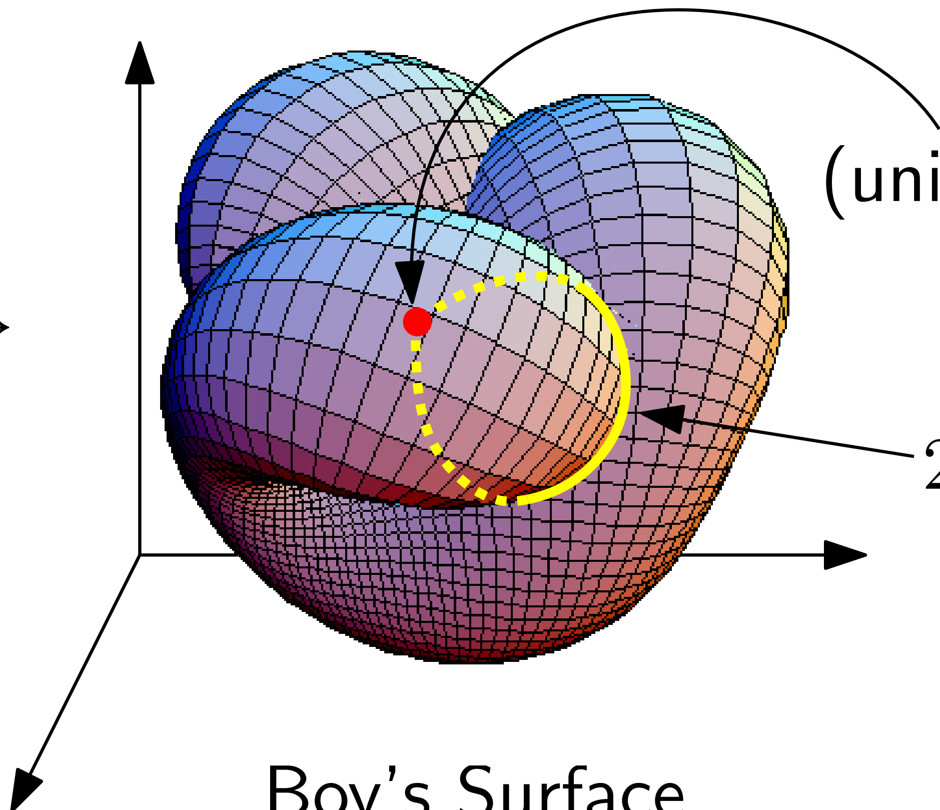

Boy's Surface

$f: \mathbb{R}P^2 \to \mathbb{R}^3$ is a 4-embedding (no 4-fold intersections)

Classical Case: Maps without 2-fold intersections

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Theorem (van Kampen–Shapiro–Wu):

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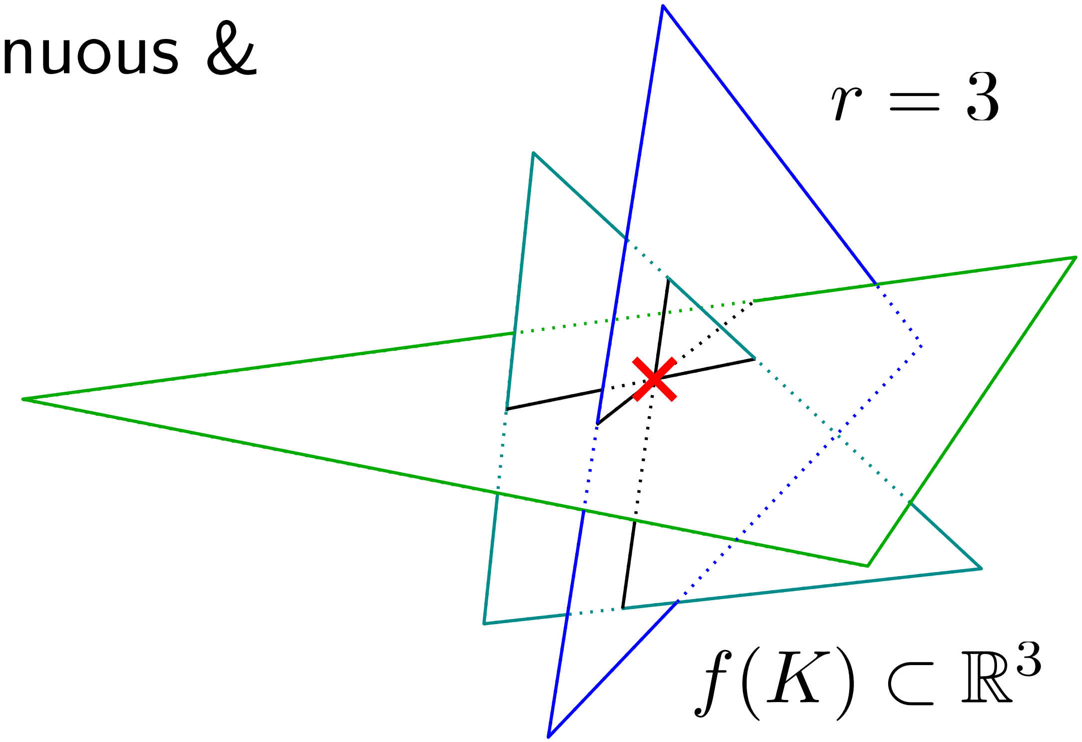

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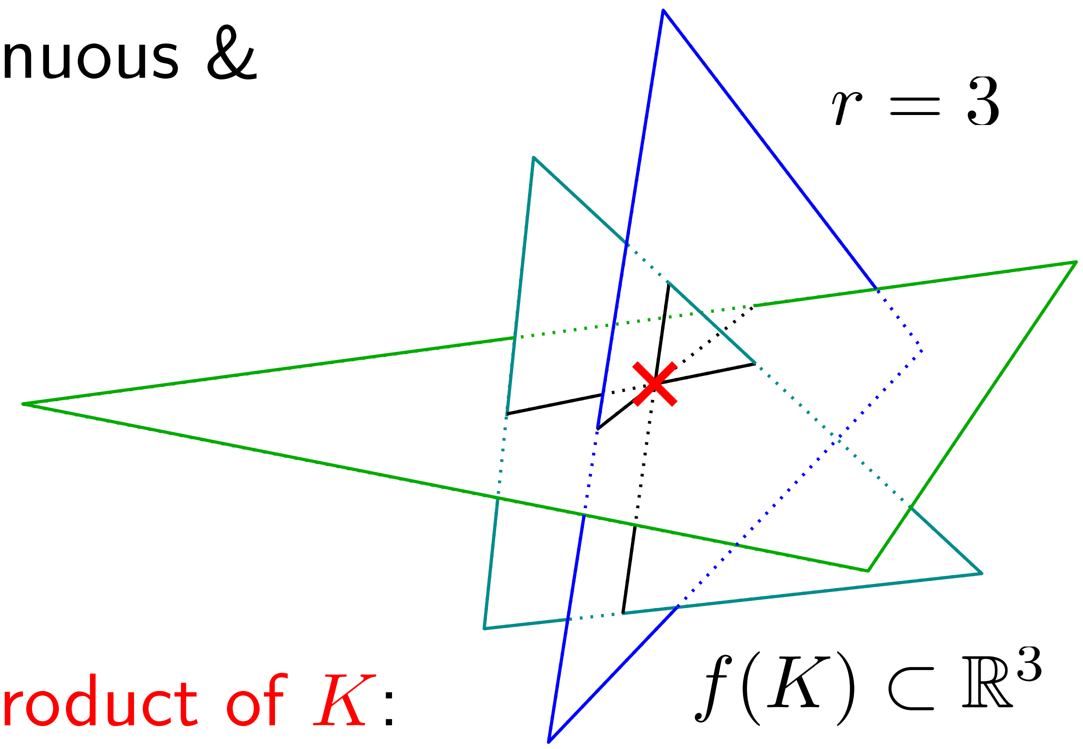

An **necessary condition**
for the existence of f:

1) Define the r-fold deleted product of K:

$$K_\delta^{\times r} := \{\sigma_1 \times \cdots \times \sigma_r \mid \sigma_i \in K \text{ and } \sigma_i \cap \sigma_j = \emptyset\} \subset K^{\times r}$$

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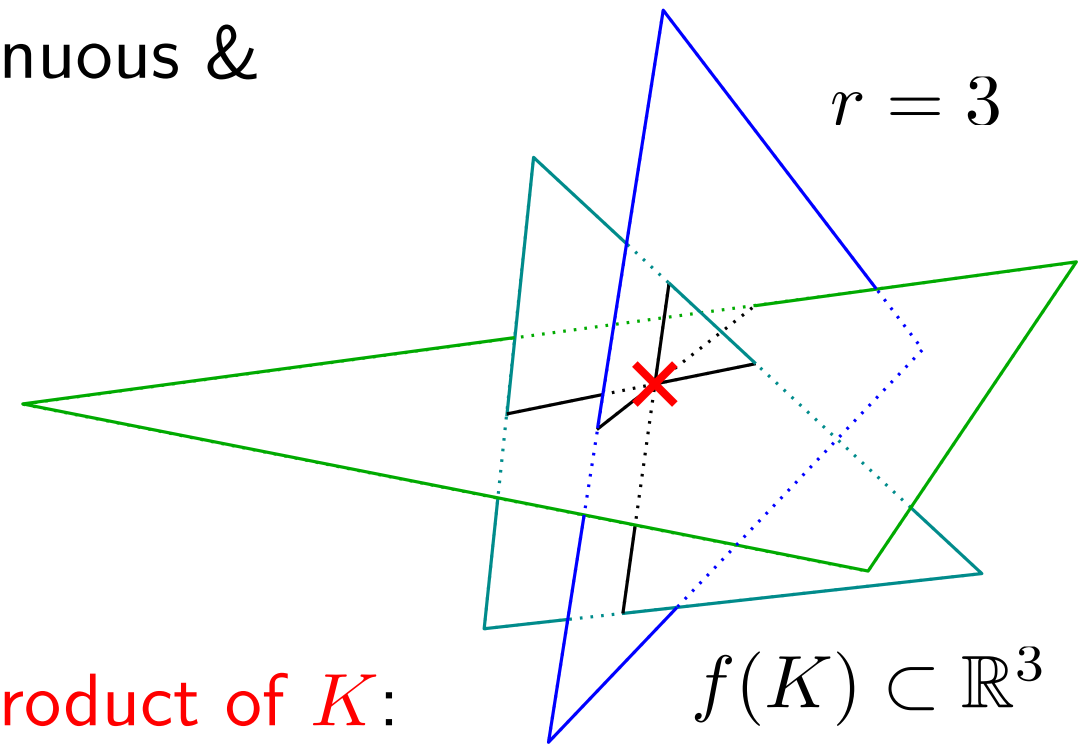

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2) Given an r-embedding $f\colon K \to \mathbb{R}^d$, define

$$\begin{array}{rccc} \widetilde{f}\colon & K_\delta^{\times r} & \to & \mathbb{R}^{d\times r} \\ & (x_1, \ldots, x_r) & \mapsto & (fx_1, \ldots, fx_r) \end{array}$$

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A) The symmetric group $\mathfrak{S}_r$ acts on both $K_\delta^{\times r}$ and $\mathbb{R}^{d\times r}$ by permutation of the coordinates

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$f\colon K^m \to \mathbb{R}^d$ such that for all $\sigma_1, \ldots, \sigma_r \in K$ with $\sigma_i \cap \sigma_j = \emptyset$

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yes!

provided $m = (r-1)k, d = rk$

and $k \geq 3$

f is an **almost r-embedding**

$f\colon K^m \to \mathbb{R}^d$ such that for all $\sigma_1, \dots, \sigma_r \in K$ with $\sigma_i \cap \sigma_j = \emptyset$

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$$\exists \widetilde{f}\colon K^{\times r}_{\delta} \to_{\mathfrak{S}_r} S^{(r-1)d-1}$$

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Corollary. The existence of $f\colon K^{(r-1)k} \to \mathbb{R}^{rk}$ almost r-embedding is *algorithmically solvable*, provided $k \geq 3$.

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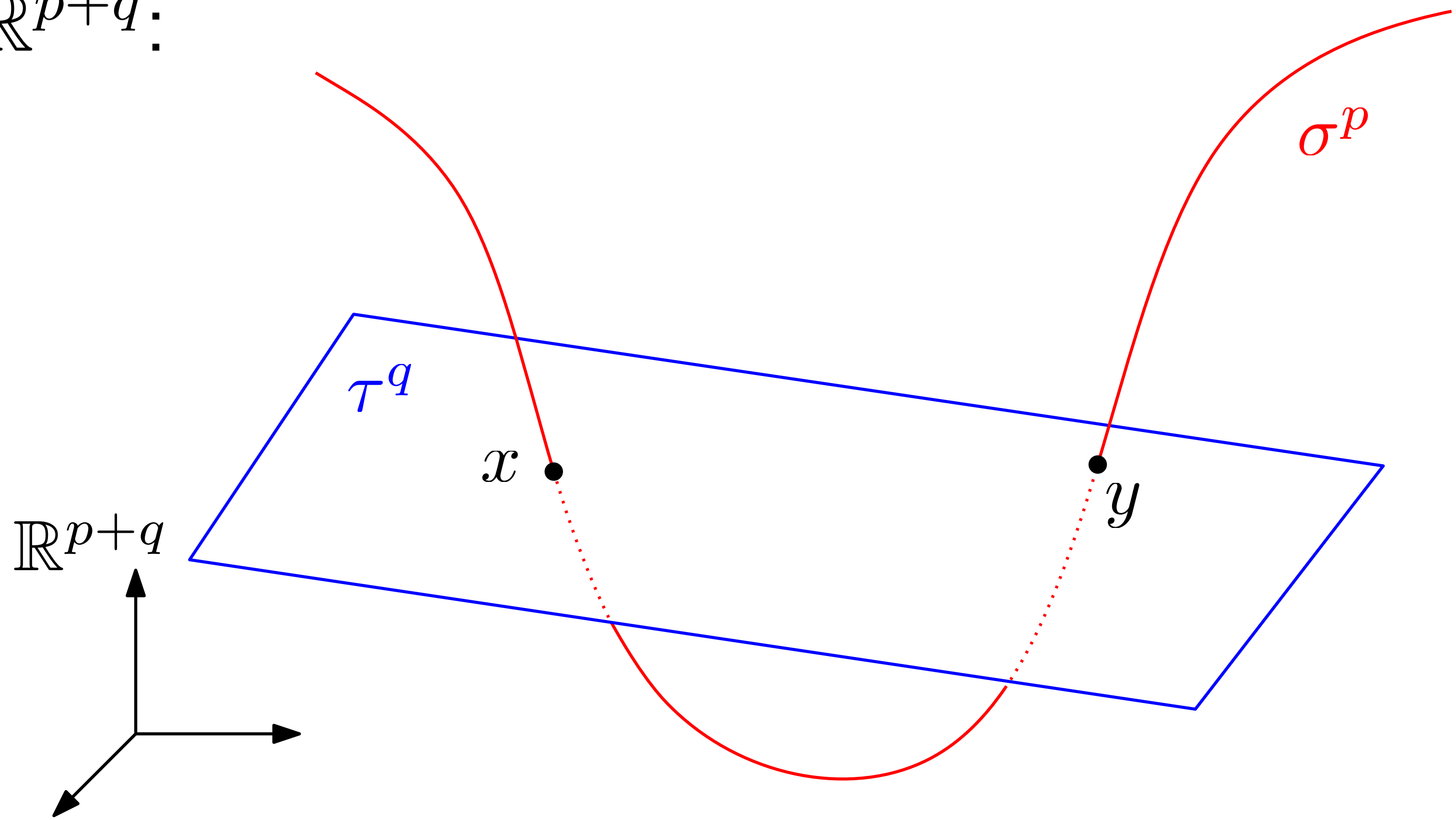
Classical Whitney Trick with two balls σ^p and τ^q in $\mathbb{R}^{p+q}$:

$p, q \geq 3$



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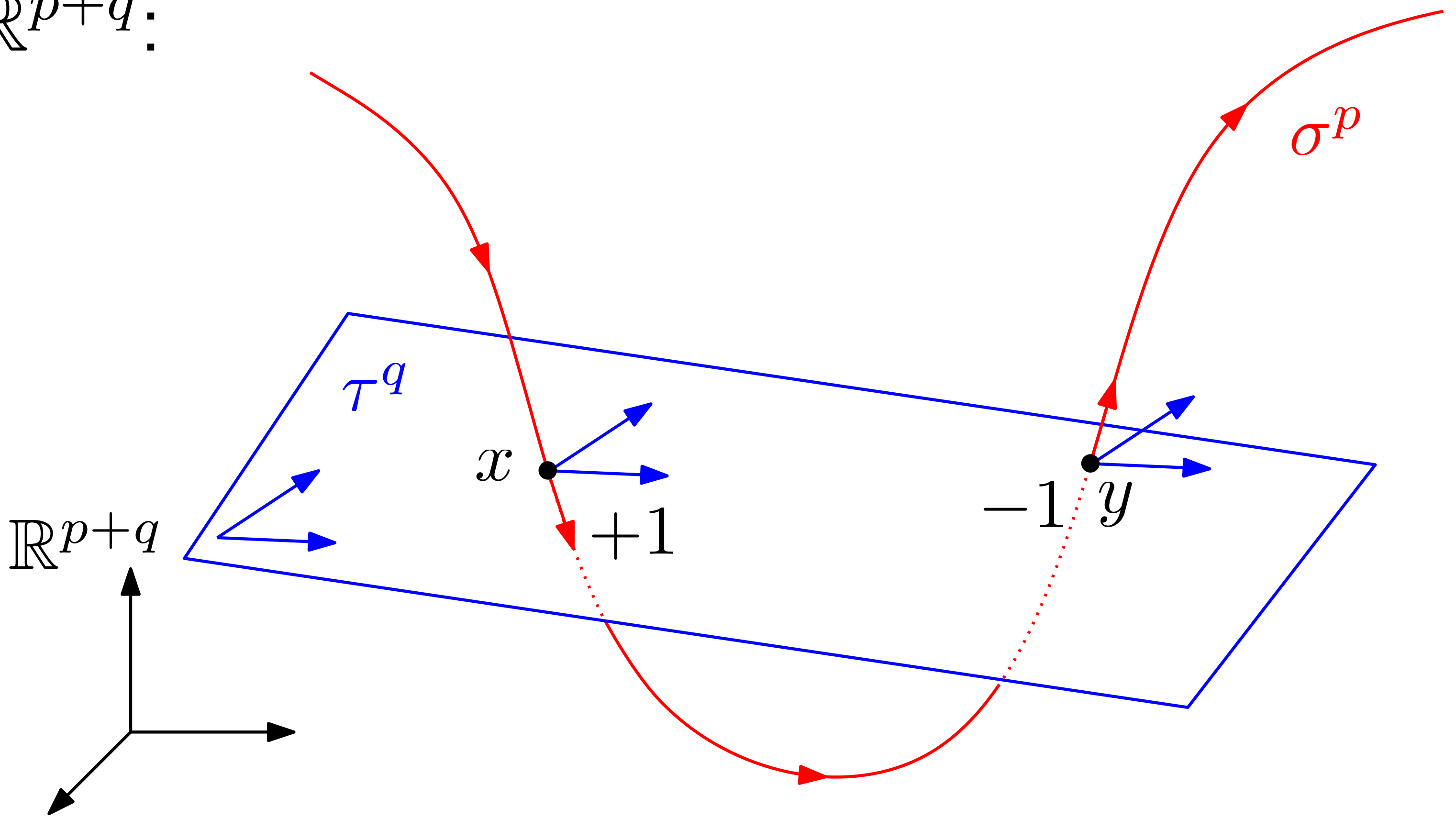

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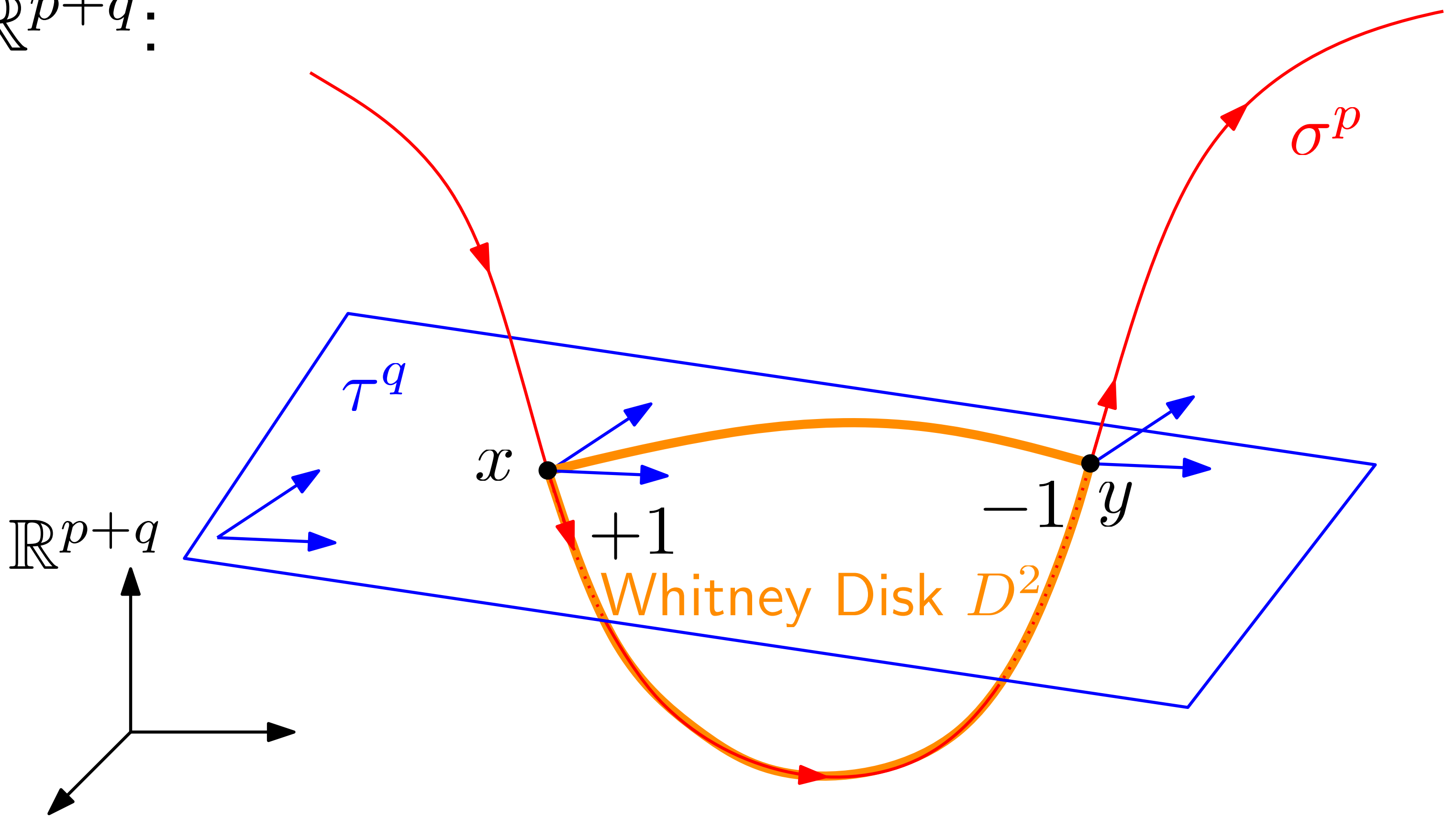

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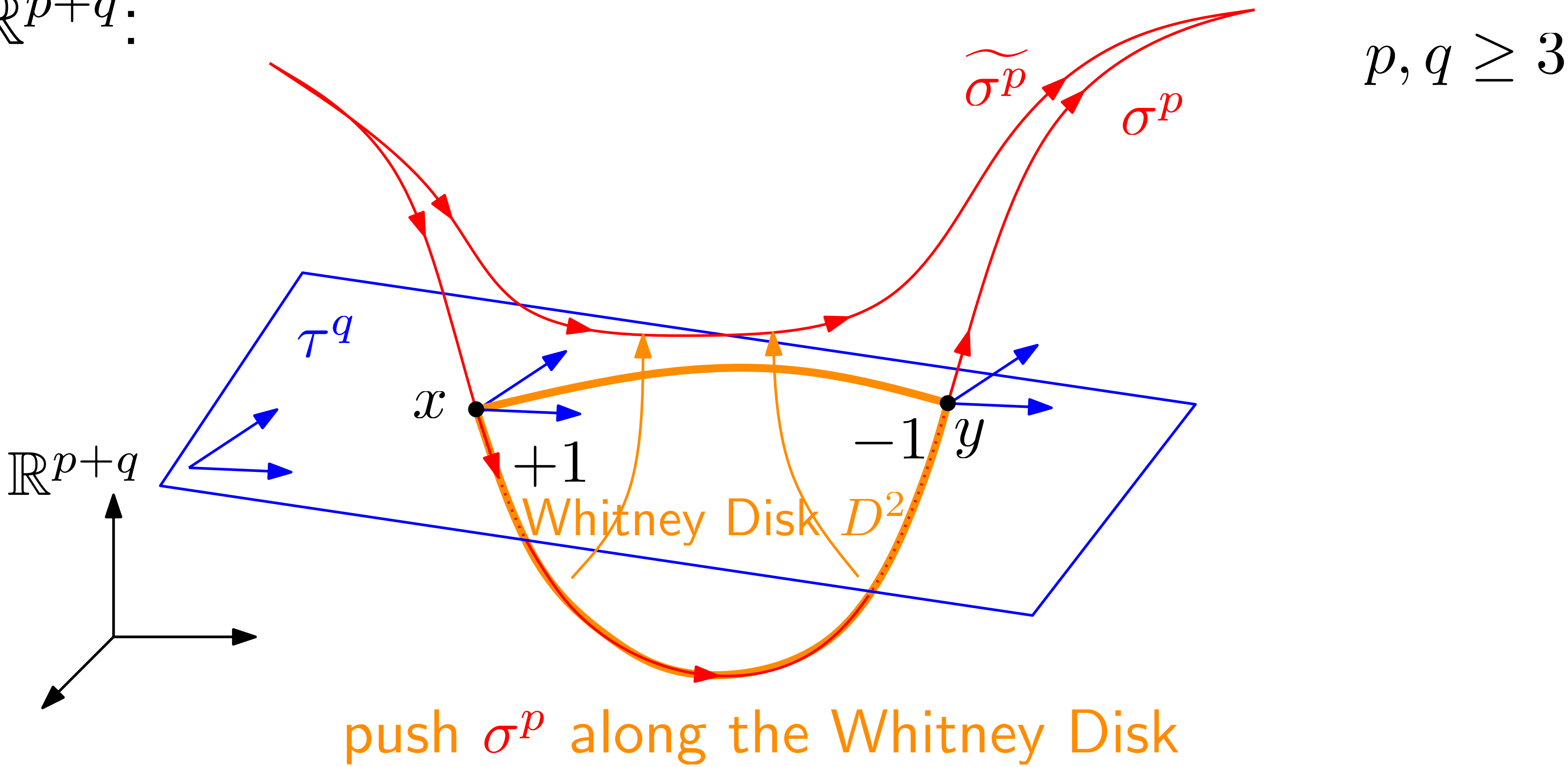
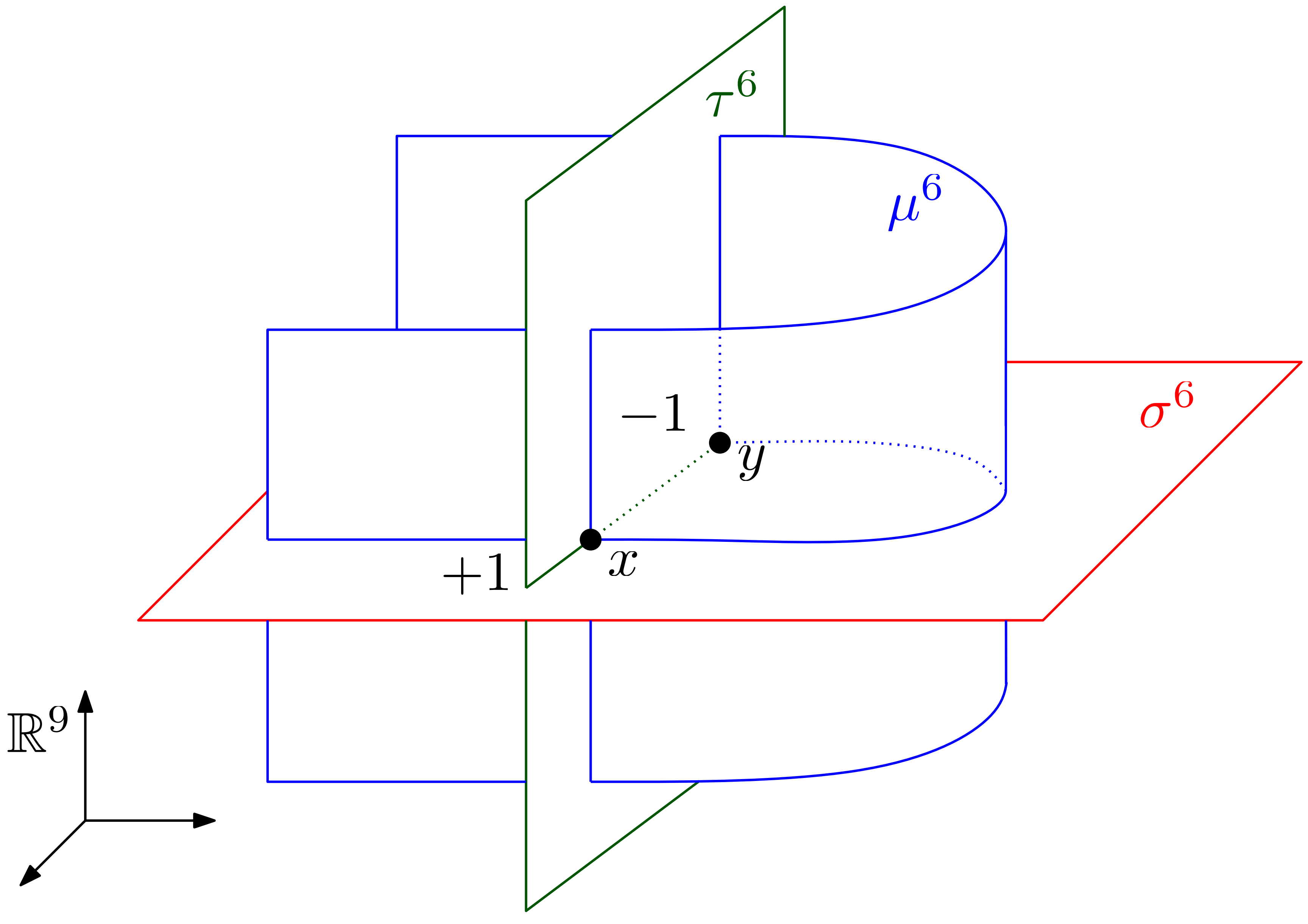


What happens with more than two balls?



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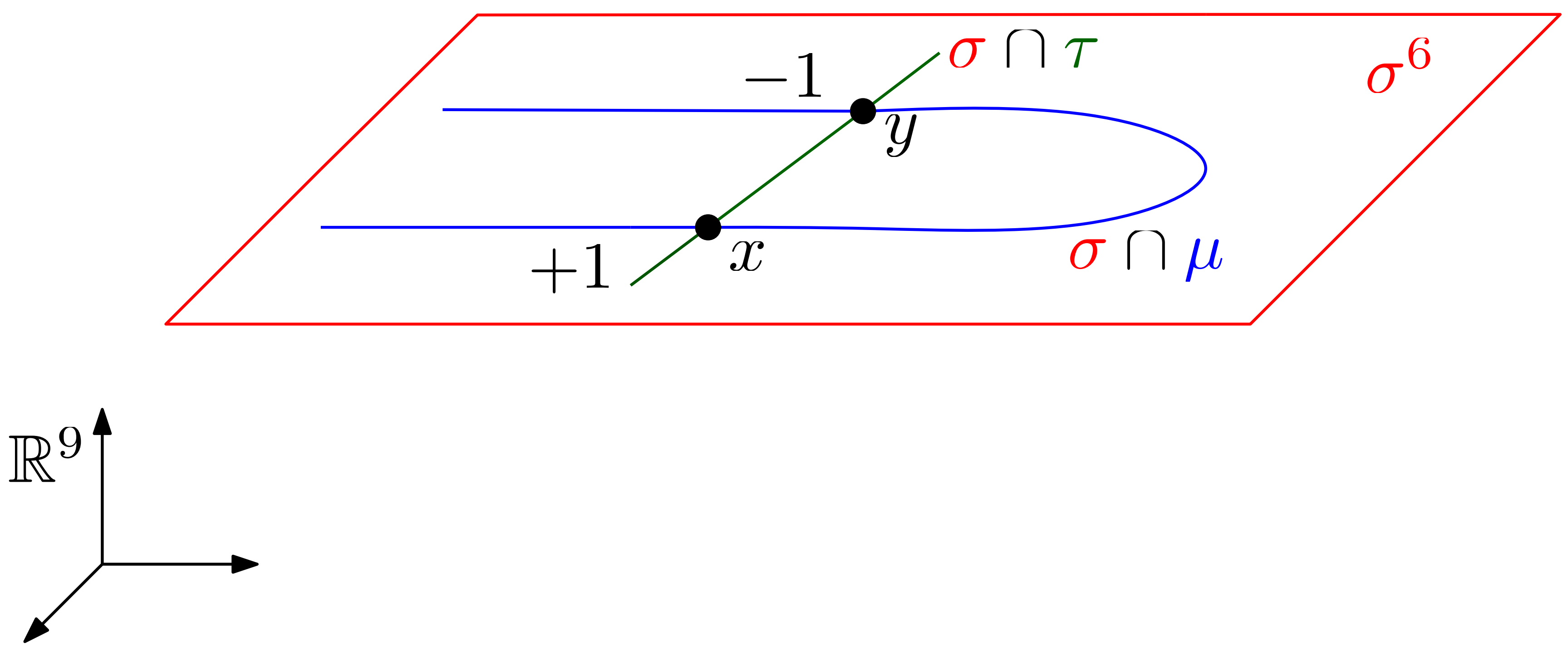

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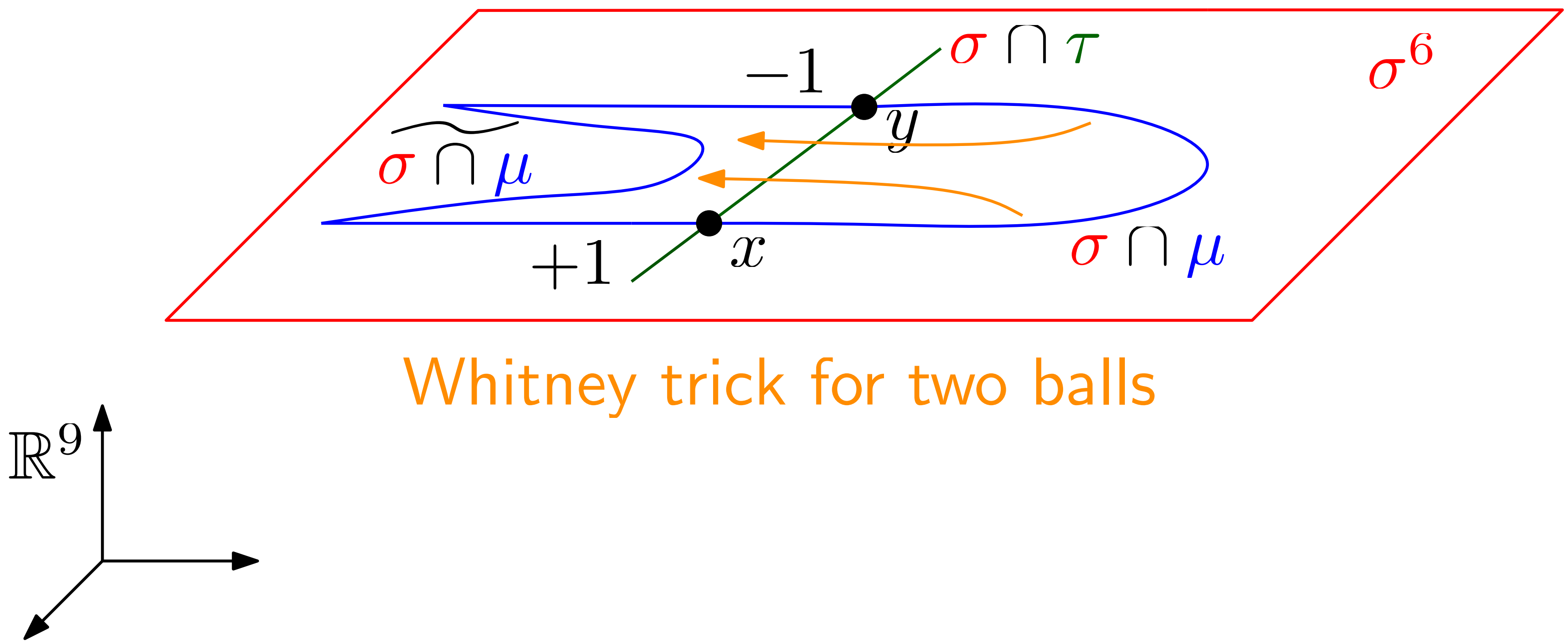

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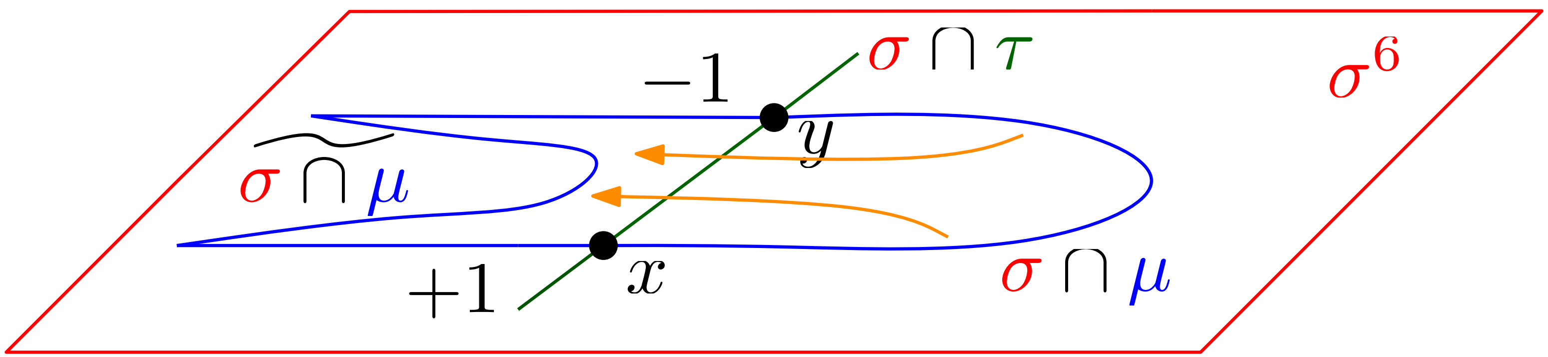
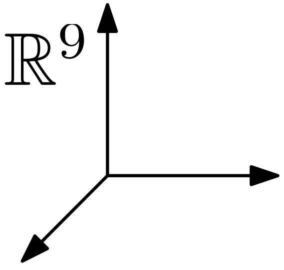


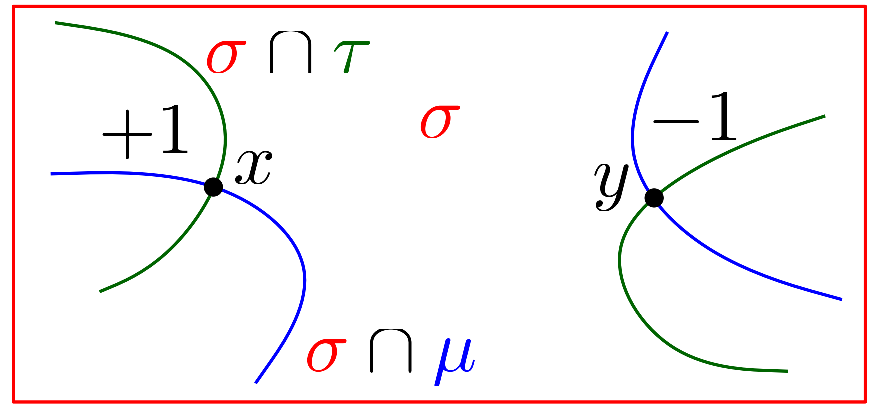
Whitney trick for two balls

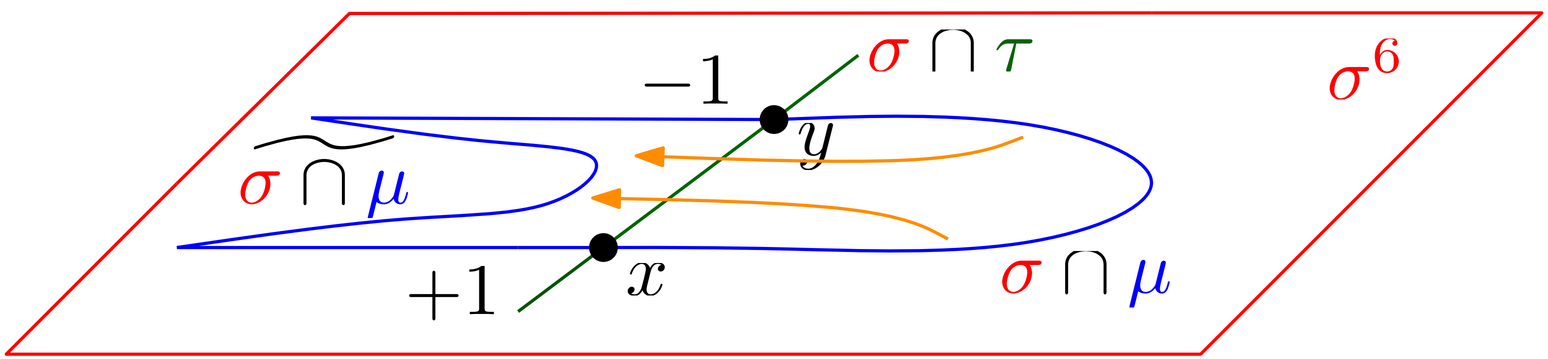


Then, use that σ^6 is "flat" (codimension ≥ 3) to extend the solution to $\mathbb{R}^9$.

What happens with more than two balls?

Problem: $\sigma \cap \tau$ and $\sigma \cap \mu$ are, in general, not connected spaces





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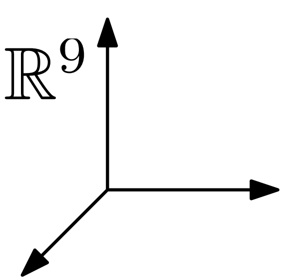

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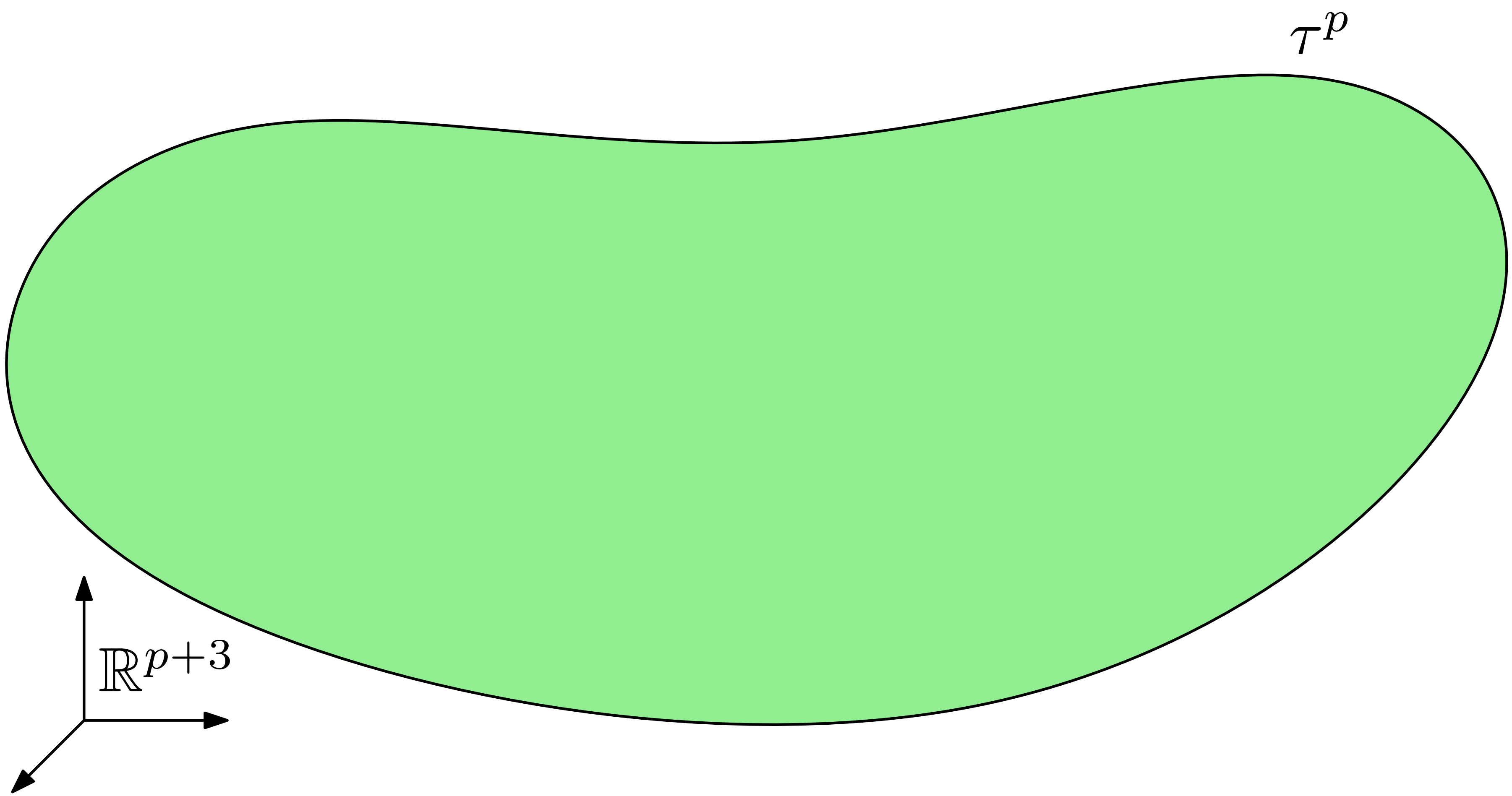

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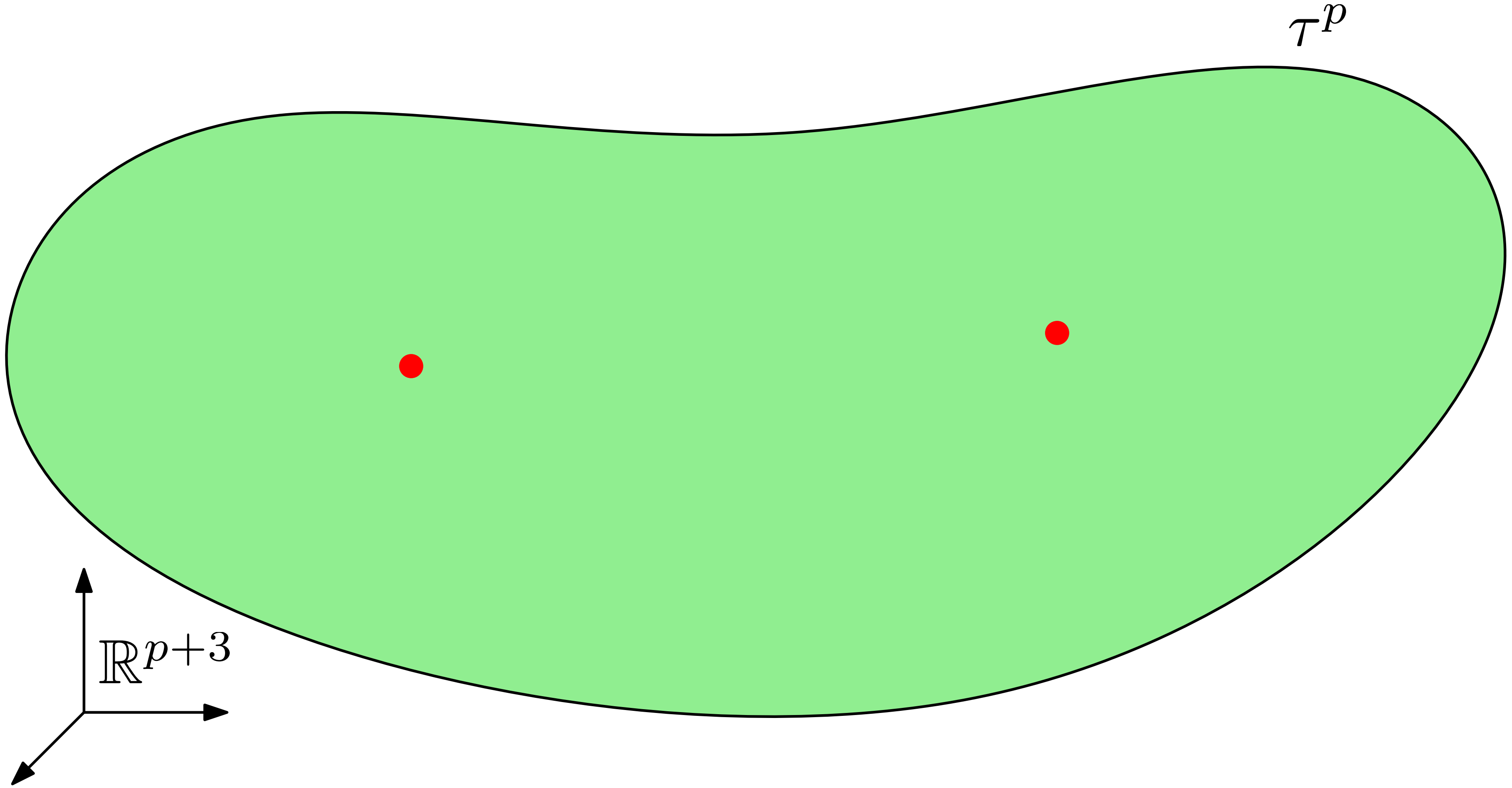

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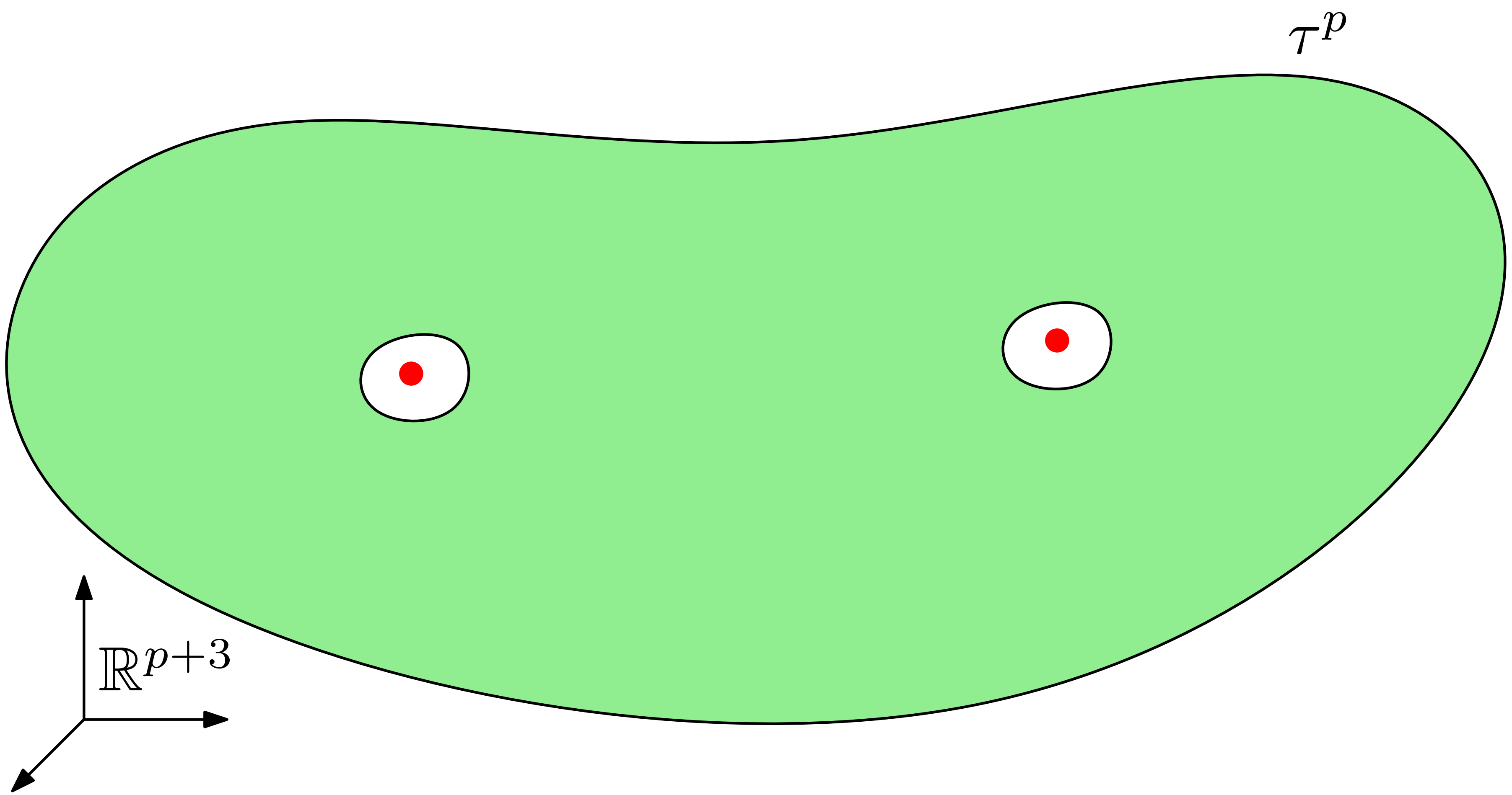

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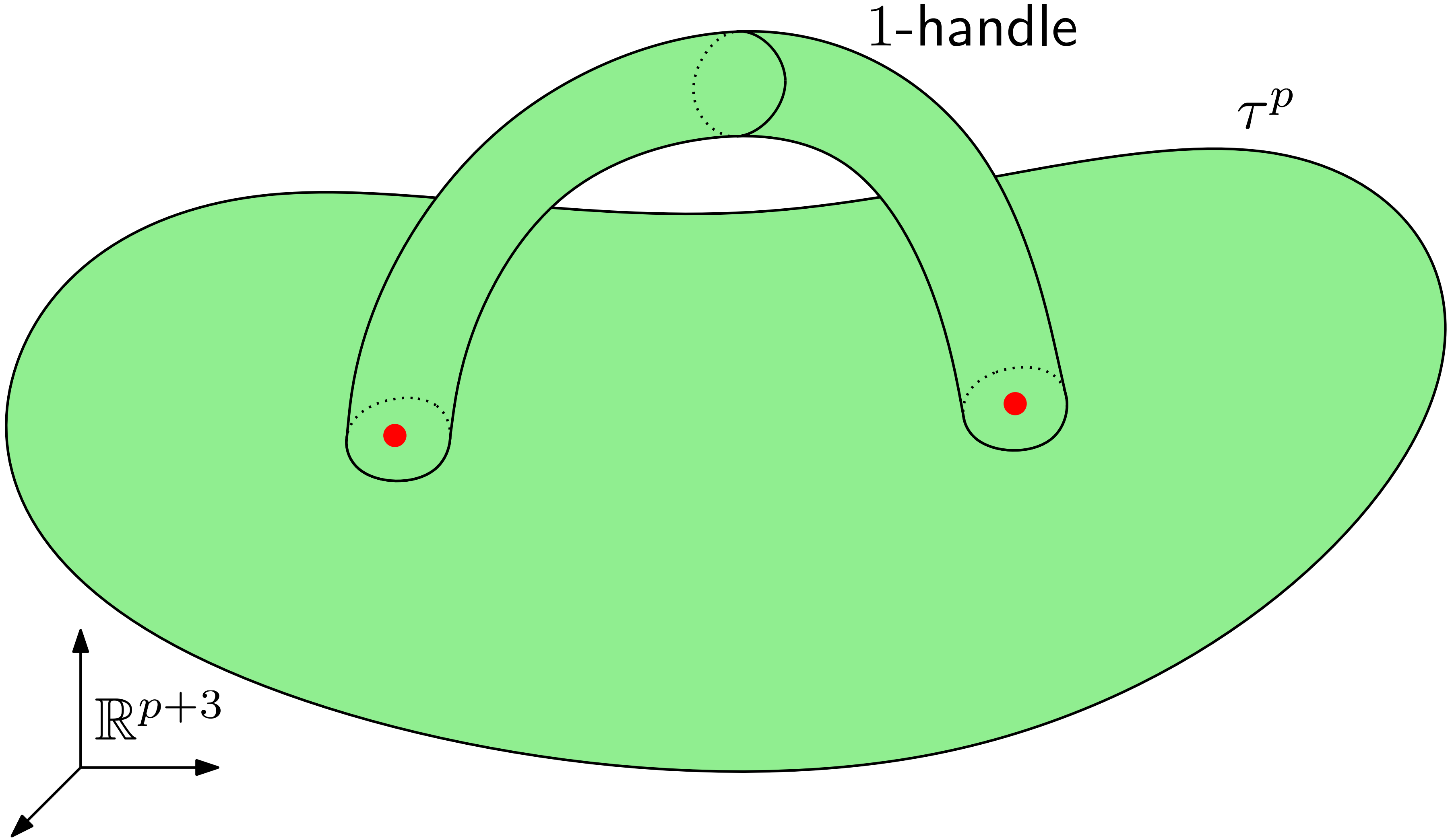

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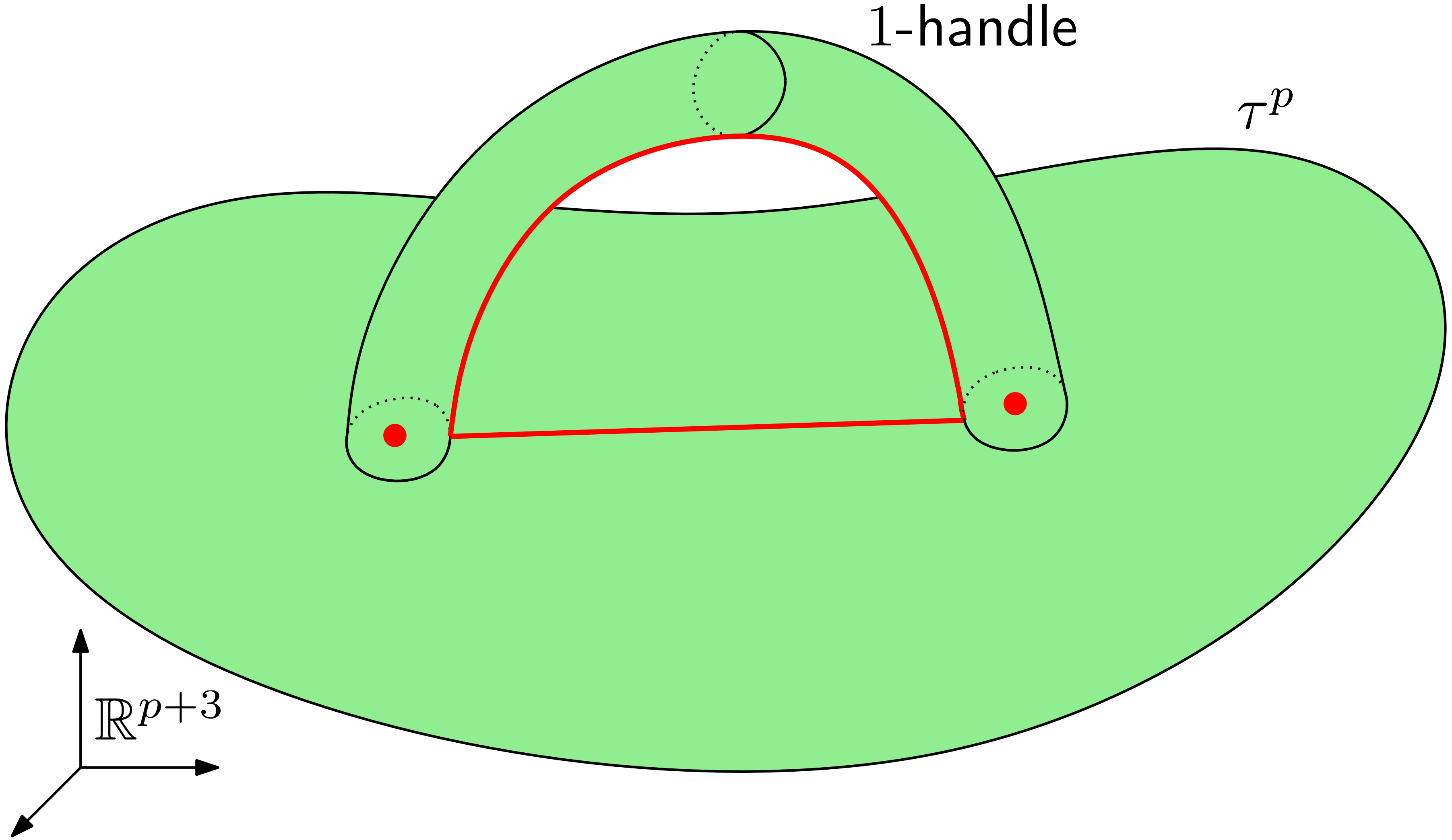

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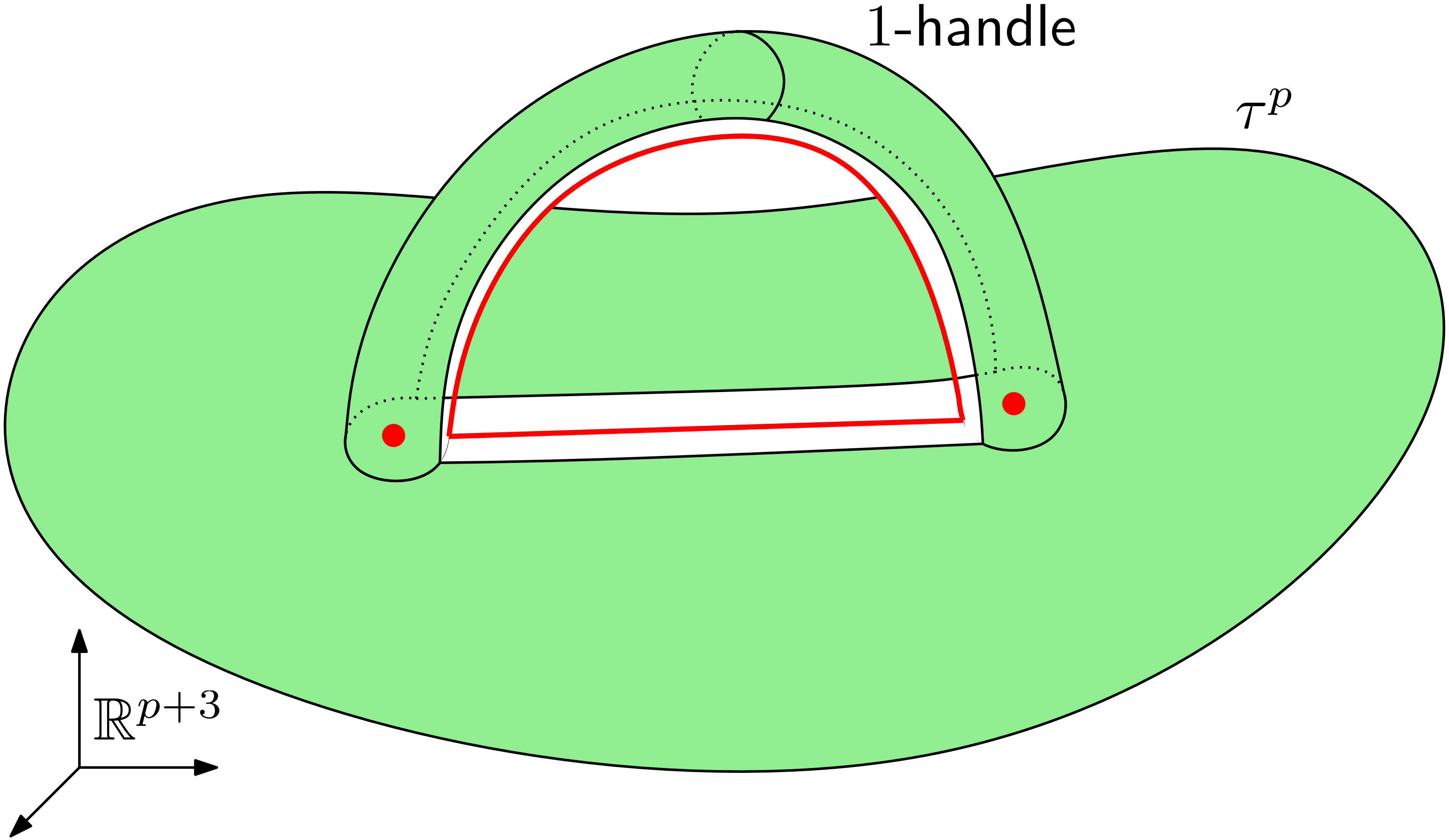

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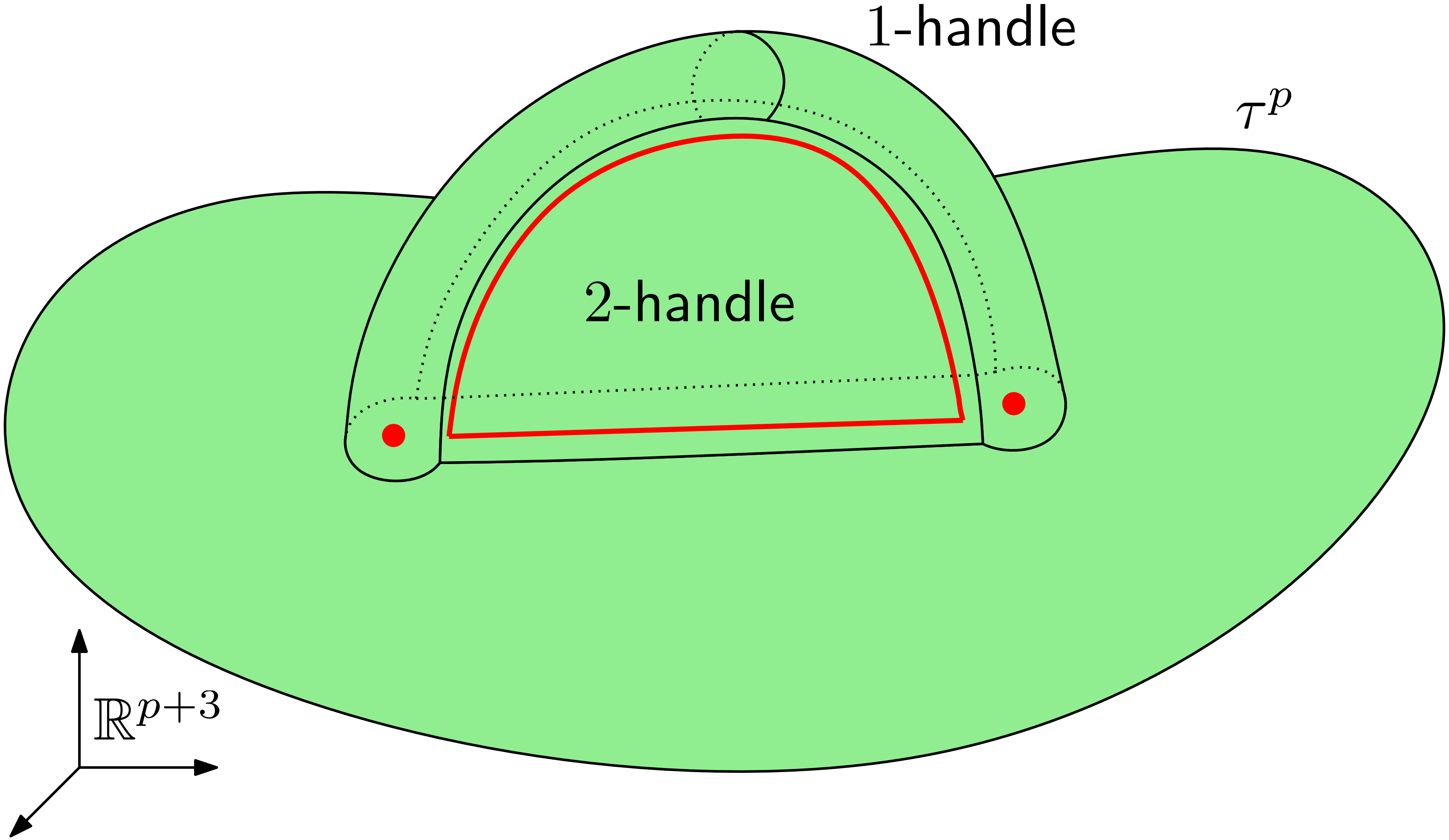

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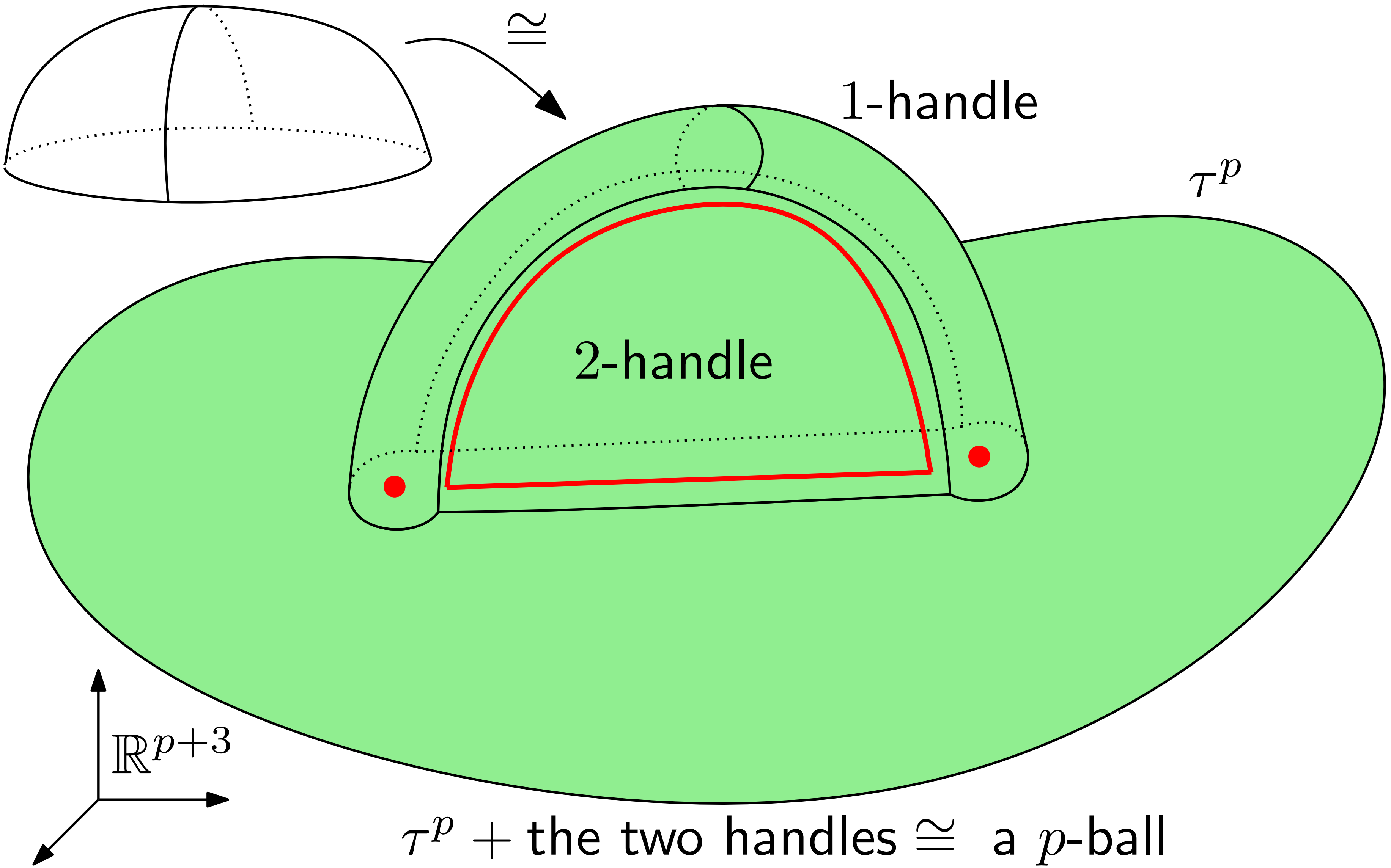
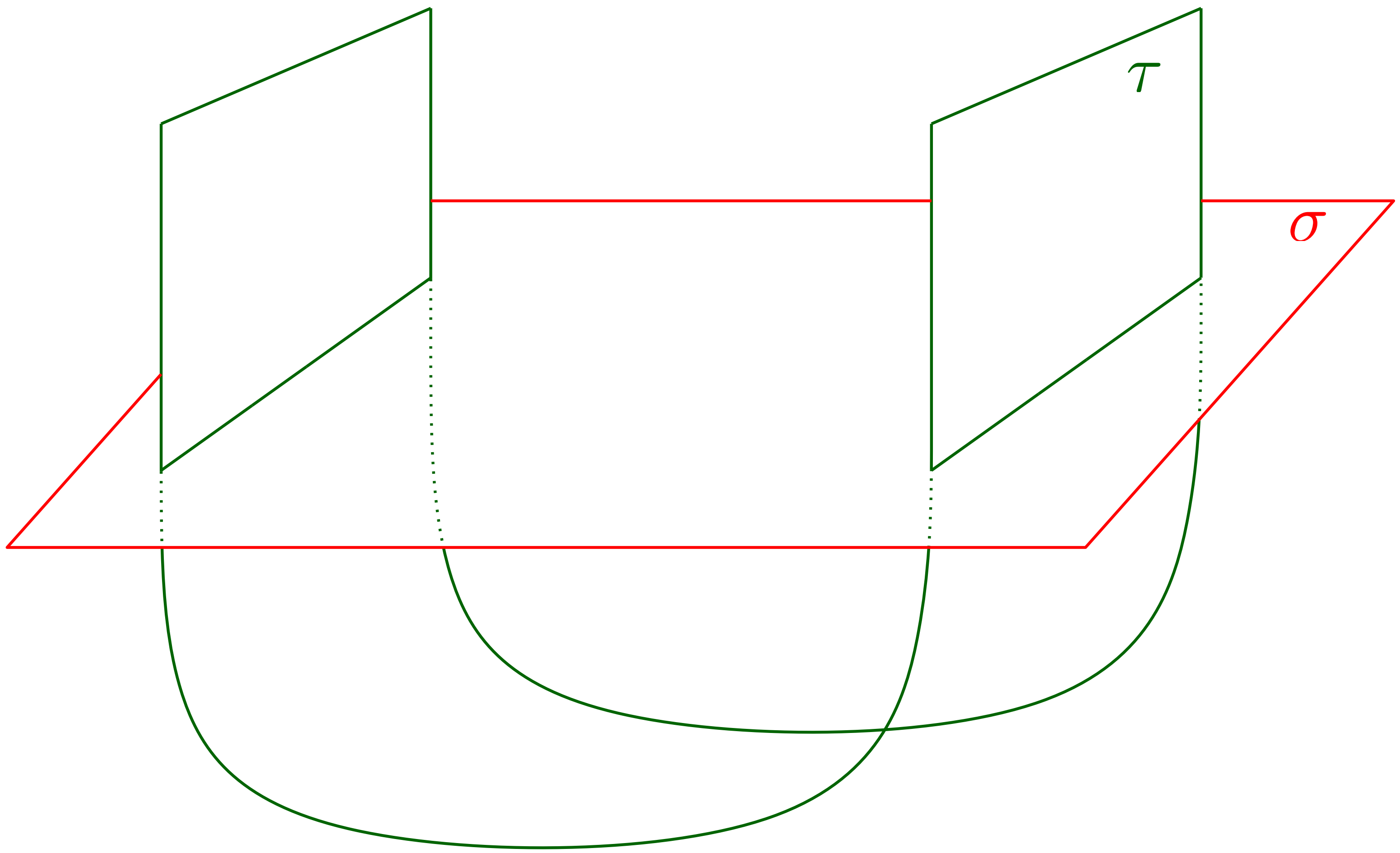


τ^p + the two handles $\cong$ a p-ball

Back to the intersection problem



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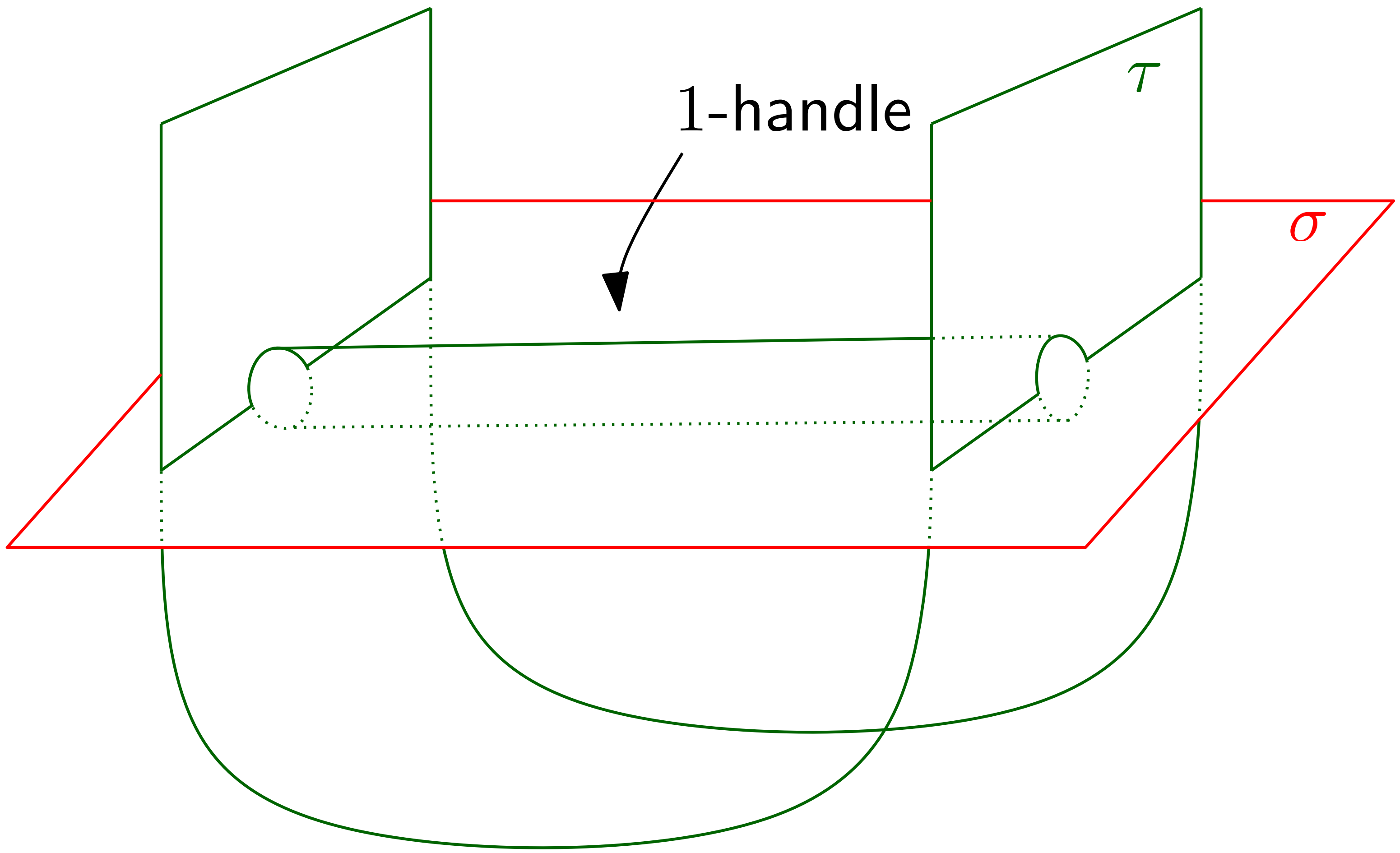

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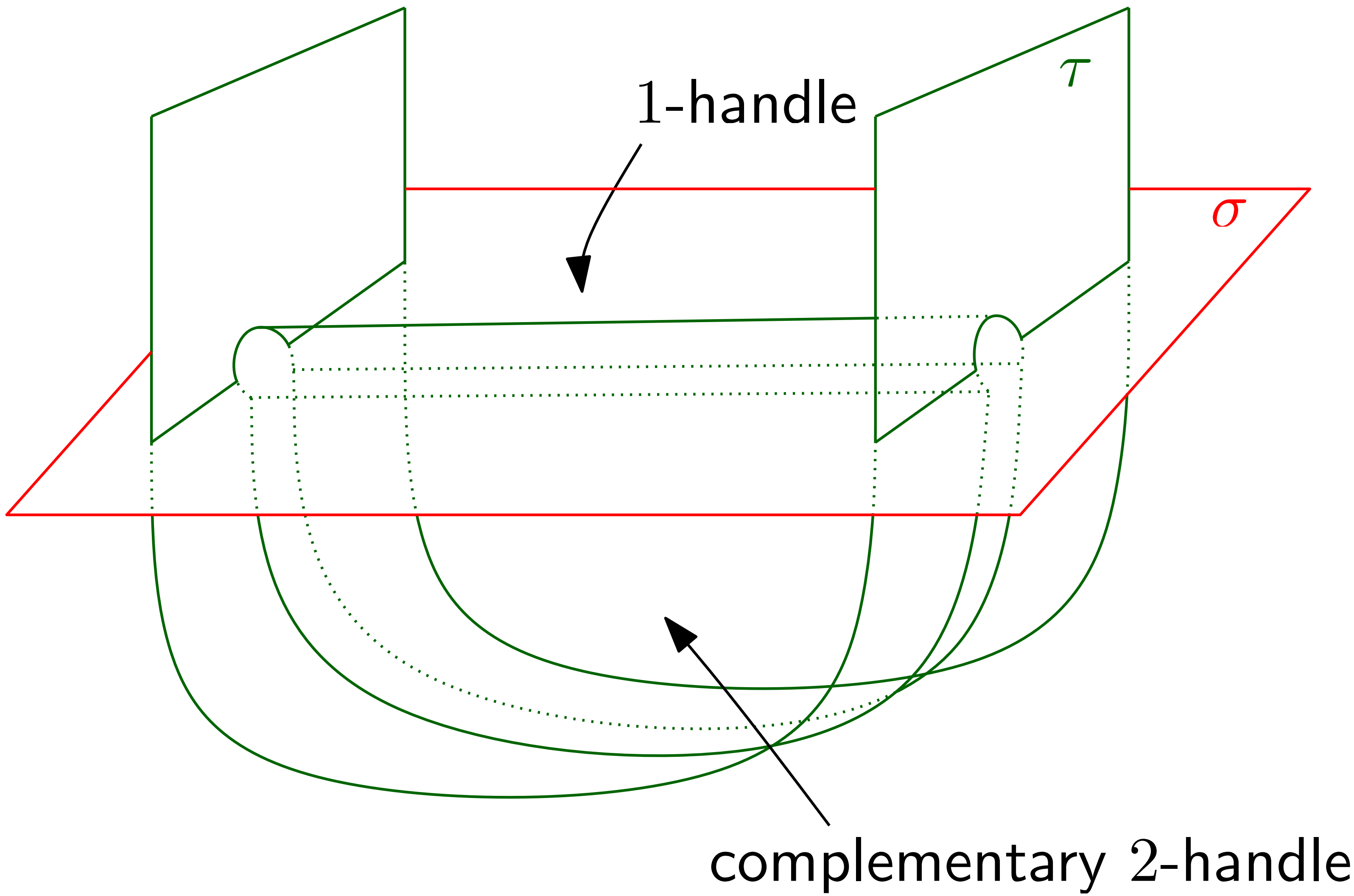

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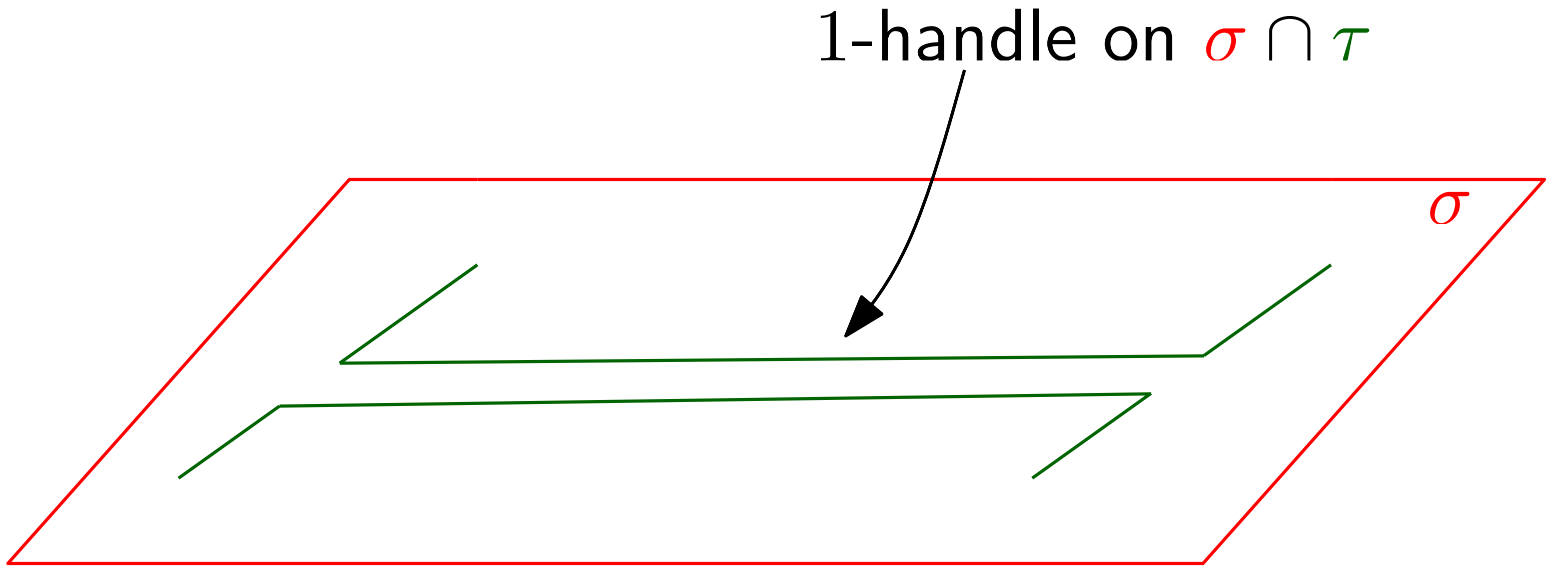
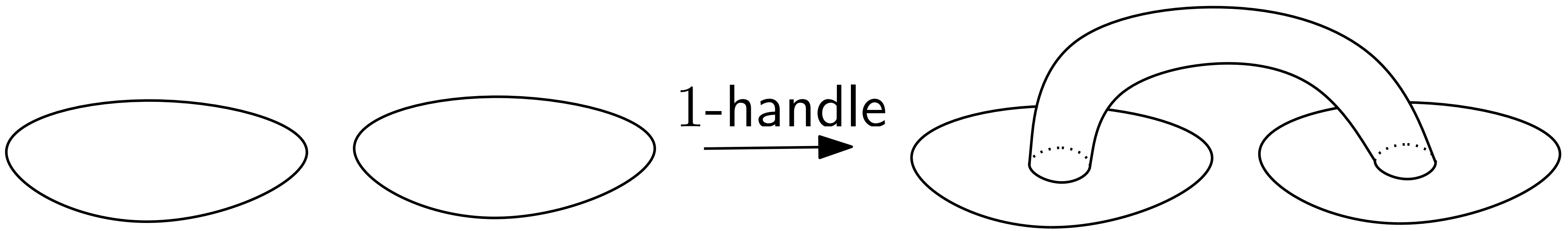
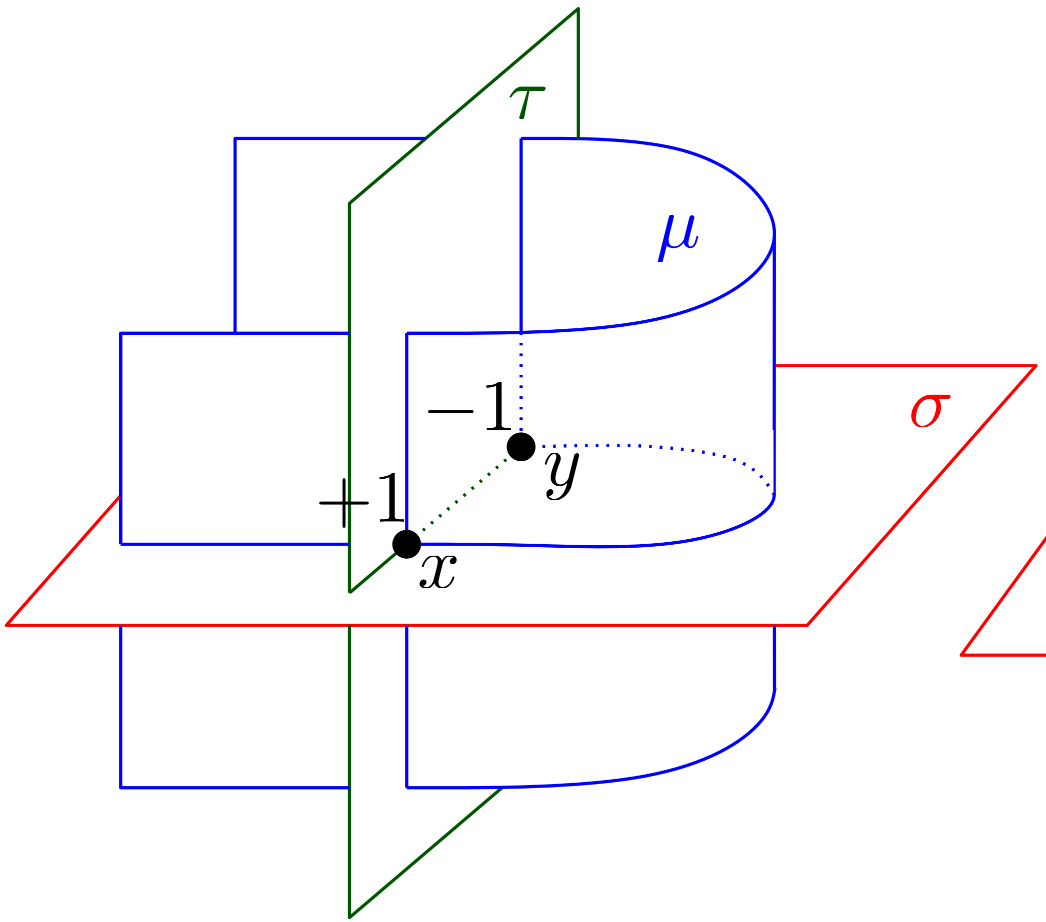
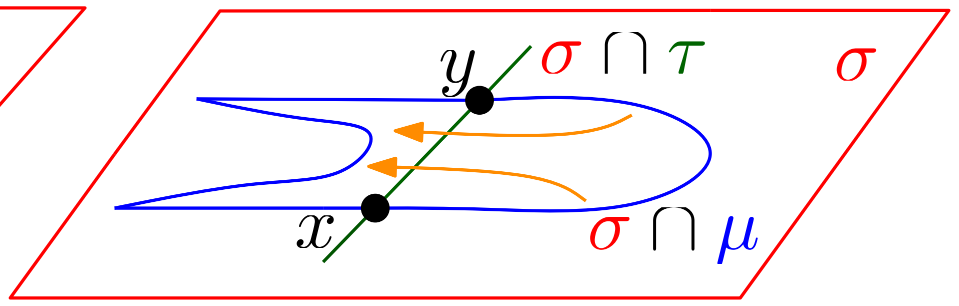


Hence, we can add 1-handles on $\sigma \cap \tau$.
I.e., we can make $\sigma \cap \tau$ connected.



3-fold Whitney Trick

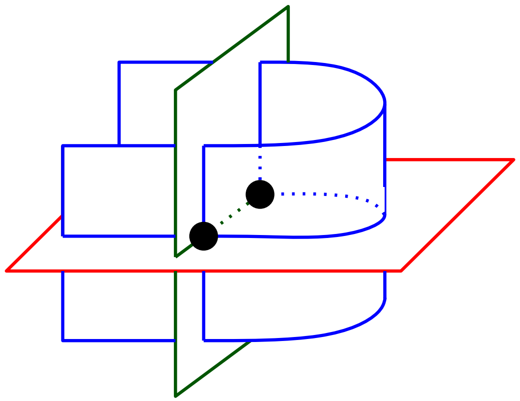




We can assume $\sigma \cap \tau$ and $\sigma \cap \mu$ are connected.
Hence we can use the classical Whitney trick to solve the 3-balls situation, i.e., to remove triple intersection points.

r-fold Whitney Trick

Given r balls $B_1, \cdots, B_r$ mapped by a f into $\mathbb{R}^d$ in general position



$$f : B_1 \sqcup \cdots \sqcup B_r \to \mathbb{R}^d$$

with

$$d - \dim(B_i) \geq 3 \quad \text{and} \quad \sum_i d - \dim(B_i) = d.$$

If

$$f(B_1) \cap \cdots \cap f(B_r) = \{x, y\}$$

two points of opposite signs. Then we can remove these two points by a move along a 2-dimensional cone ($\approx$"Whitney disk").
In particular, we can avoid any codimension ≥ 3 object in $\mathbb{R}^d$ during this move.

classical Whitney Trick $\Rightarrow$ first part of Van Kampen Embeddability ($k \neq 2$):

$$K^k \to \mathbb{R}^{2k} \text{ almost } 2\text{-embeds}$$
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r-fold Whitney Trick $\Rightarrow$ For $r, k \geq 3$,

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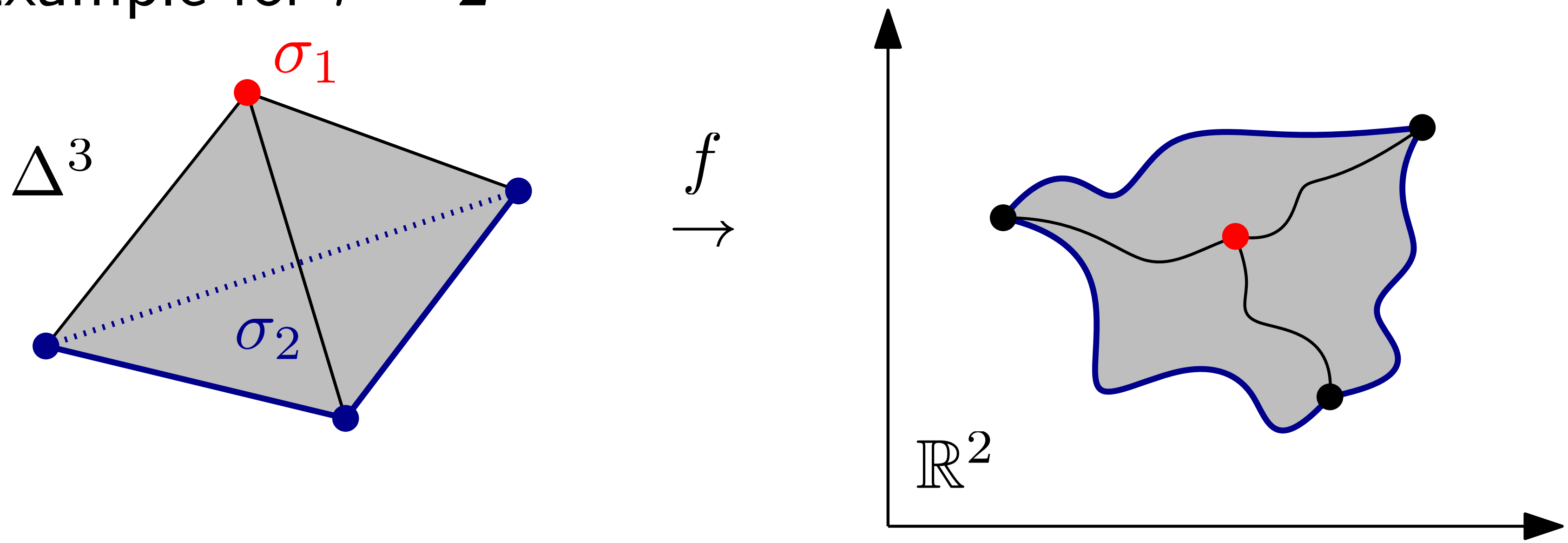
check a system of linear equations over $\mathbb{Z}$

Application: Topological Tverberg

Topological Tverberg Conjecture: Given $r, d \geq 2$, there exists **no** almost r-embedding

$$\Delta^{(r-1)(d+1)} \to \mathbb{R}^d.$$

Example for $r = 2$



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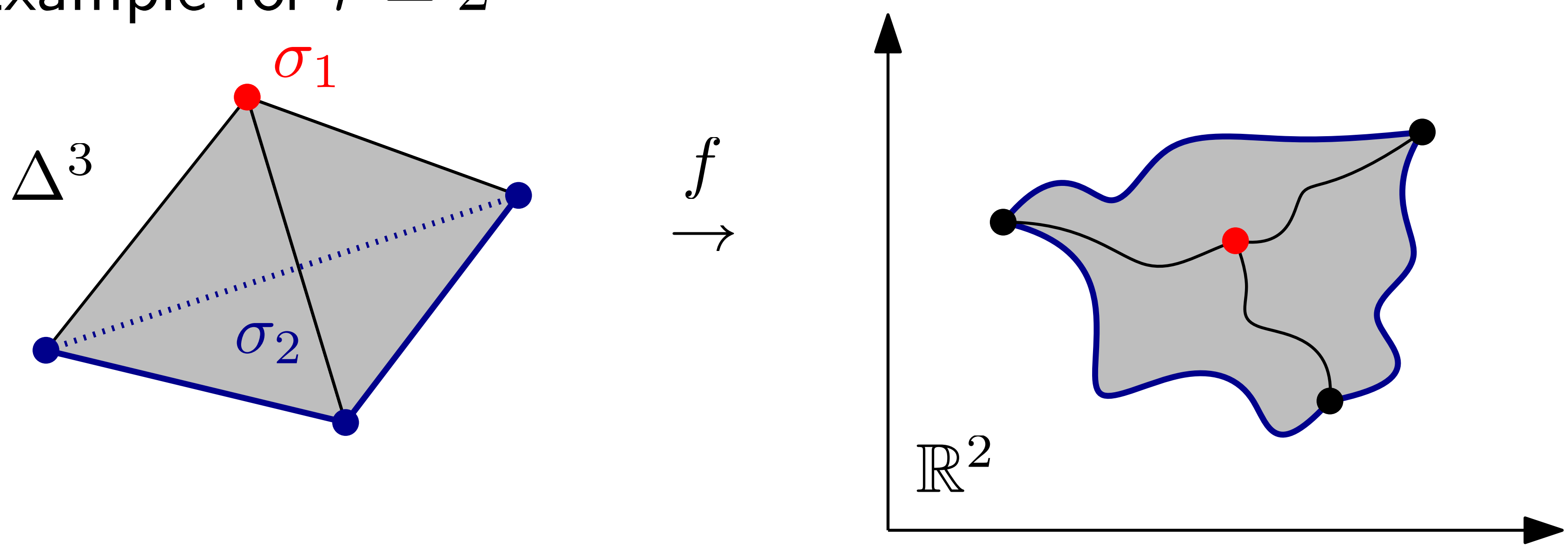

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The conjecture holds for $r = \text{prime}^{\text{power}}$ (Ozaydin87)

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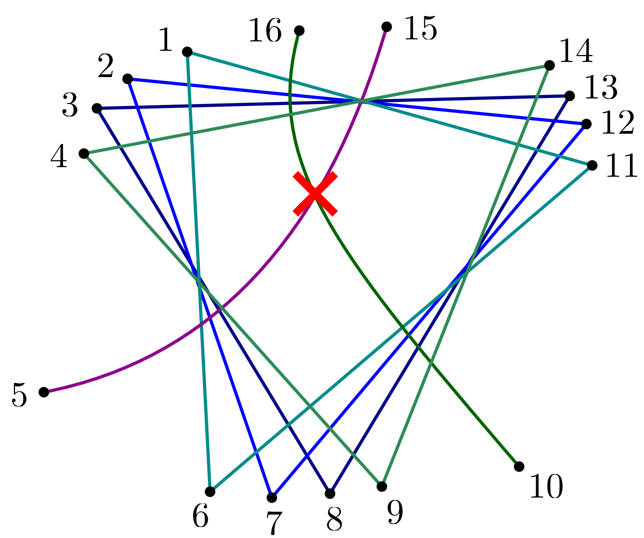
First open case of the conjecture: almost 6-embedding $\Delta^{15} \to \mathbb{R}^2$. I.e., a drawing of K_{16} without



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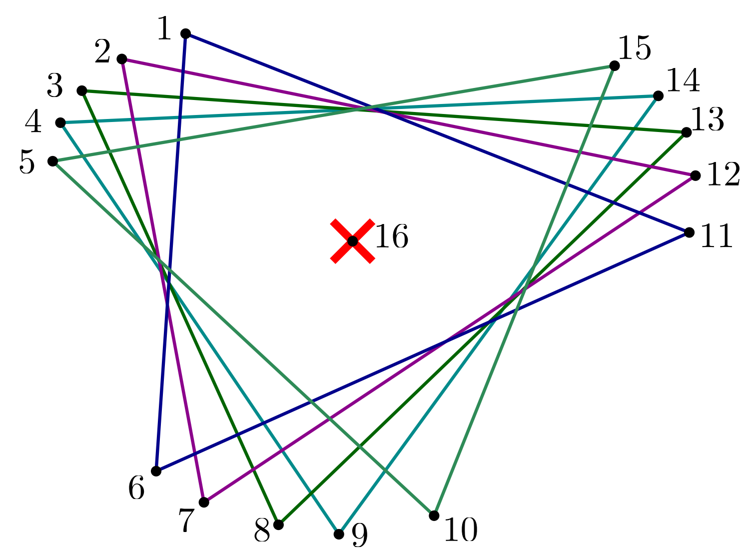

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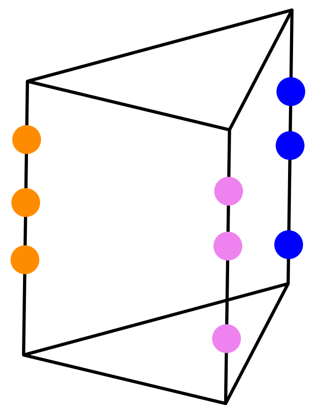
$\Delta^8 \to$  $\subset \mathbb{R}^3$

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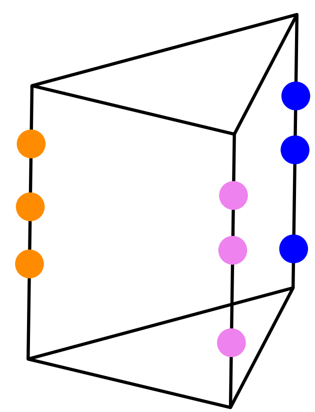
$\Delta^8 \to$  $\subset \mathbb{R}^3$

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2) A *codimension 2* (!) Whitney Trick

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M-Wagner r-fold Whitney Trick

Gromov trick, Constraint method (BFZ)

M-Wagner prismatic counterexample $\Delta^{95} \to \mathbb{R}^{18}$

$$18 = 6 \cdot 3 + 0$$

- 6: Ozaydin (smallest non-prime power)
- 3: M-Wagner r-fold Whitney Trick
- 0: prismatic maps

Avvakumov-M-Skopenkov-Wagner prismatic codim 2 counterexample $\Delta^{65} \to \mathbb{R}^{12}$

$$12 = 6 \cdot 2 + 0$$

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What happens in lower dimension $(2 \leq d \leq 11)$ remains a mystery...

In another direction...

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$$\exists f : K^m \hookrightarrow \mathbb{R}^d \quad \Leftrightarrow \quad \exists \widetilde{f} : K_\delta^{\times 2} \to_{\mathfrak{S}_2} S^{d-1}$$

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Corollary. The existence of an embedding $K^m \hookrightarrow \mathbb{R}^d$ is *algorithmically solvable*, provided $d \gtrsim 1.5m$

Theorem (M-Wagner)

$\exists f\colon K^m \to \mathbb{R}^d$ almost r-embedding $\Leftrightarrow$ $\exists \widetilde{f}\colon K_\delta^{\times r} \to_{\mathfrak{S}_r} S^{(r-1)d-1}$

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