Representation Stability in Configuration Spaces via Whitney Homology of the Partition Lattice

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(based on paper to appear in IMRN)
A “Point” in a Configuration Space with $S_n$-rep’s on Cohomology

- Manifold = 3-holed torus
- $\eta = 6 = \# \text{distinct labeled points}$
- $S_n$ acts freely on configuration space by permuting pt. labels, inducing rep’n on each cohomology group
**Representation Theoretic Stability**

**Defn (Church, Farb):** A series of $S_n$-modules $M_1, M_2, \ldots$ for $n=1,2,\ldots$ stabilizes at $B>0$ if for each $n>B$, we have

$$M_n = \sum c_\lambda V(\lambda)$$

where $V(\lambda) = S^{(n-m, \lambda)}$ for $\lambda + m \leq B$

and where $c_\lambda$ does not depend on $n$

**E.g.**

$$M_n \rightarrow \begin{array}{c} \text{Diagram 1} \\ n-m \end{array} + \begin{array}{c} \text{Diagram 2} \end{array} \

\rightarrow \begin{array}{c} \text{Diagram 3} \end{array} + \begin{array}{c} \text{Diagram 4} \end{array}$$

$$M_{n+1} \rightarrow \begin{array}{c} \text{Diagram 5} \end{array} + \begin{array}{c} \text{Diagram 6} \end{array}$$

$$M_{n+2} = \begin{array}{c} \text{Diagram 7} \end{array}$$

**Our Focus:** $S_n$-reps from partition lattice
Our Starting Point:

*Thm (Church-Farb):* \( H^i(M_n, \Omega) \) stabilizes for \( n \geq 4i \) where \( M_n \) is configuration space of \( n \) distinct points in plane \& \( i \) is held fixed.

*Thm (Church-Farb):* More generally, letting \( M_n^d \) be the configuration space of \( n \) distinct labeled points on connected orientable \( d \)-manifold, \( H^i(M_n^d, \Omega) \) stabilizes for  
\[
\begin{cases} 
  n \geq 4i & \text{if } d = 2 \\
  n \geq 2i & \text{if } d > 2 
\end{cases}
\]

Our First Objective:

Sharpen these bounds for \( M_n^d = R^d \)
How Representation Stability Typically Arises

- Finite number of irreducible rep's $S^\lambda; S^\lambda$ 1st appearing in $M_{|\lambda|}$
- Each $M_n$ with $n \geq |\lambda|$ likewise includes $S^\lambda \otimes \text{triv} \uparrow^{S_n}_{S_{|\lambda|} \times S_{n-|\lambda|}}$
- Church-Ellenberg-Farb prove stability bounds of $n = 2 \max |\lambda|$
- H-Reiner prove sharp stability bounds for $\text{PConf}(\mathbb{R}^d)$ at $n = \max (|\lambda| + 1, )$

**Pieri Rule:**

$$\begin{align*}
S^\lambda \otimes \text{triv} & \uparrow^{S_n}_{S_{|\lambda|} \times S_{n-|\lambda|}} \\
\text{} & = \\
\uplus & \end{align*}$$
Church-Farber Method for Orientable Manifolds

- Use Totaro's $E_2$-page of Leray spectral sequence (showing cohom. of config. space of $n$ distinct pts on manifold $M$ is determined by cohom. of $M + H^c(M_n(R^d))$) to deduce stability of each page from previous page:

$$E_2^{p, q} = \bigoplus H^p(d-1) \left( C_{S(R^d)} \right) \otimes H^q(M_S)$$

S with $|S| = n-8$ product of subspace arrangement complements

for set partition $S$ with 15 parts

E.g. for $S = \{1,3\} \cup \{2,4,5\}$

- $C_{S(R^d)} = \{ x \in (R^d)^5 | x_1 \neq x_3; x_2 \neq x_4 \}$
- $C_{S(R^d)} = C_{\{1,3\}}(R^d)^2 \times C_{\{2,4,5\}}(R^d)^3$
- $M^S = \{ x \in M^5 | x_1 = x_3; x_2 = x_4 = x_5 \}$

$E_2^{p, 8} = 0$ for $d-1 \geq 8$
Partition Lattice $\tau(n)$ & its $S_n$-representations

- $S_n$ acts by permuting values
  - e.g. $(13)[12|3|45] = 321|1|45$
Reinterpreting via Subspace Arrangement Complements

- $M_n$ = complement of type A (complex) braid arrt $\{x_i=x_j \mid 1 \leq i < j \leq n\}$

**Warning:**

figure is IR-picture, need C-picture

(Config space pt $p_i \leftrightarrow x_i \in \mathbb{C}$)

- $\Pi_n = \text{intersection poset } \Delta(A_{n-1})$

- $S_n$-module structure for $H^*(M_n)$ will translate to "Whitney homology" in $\Pi_n$, $WH_i(\Pi_n)$
\( \mathcal{L}(A_2) = \)

\( \text{poset of intersections of subspaces} \)

\[ l_1 \cap l_2 \cap l_3 \]

\( \mathbb{R}^2 = \text{empty intersection} \)

\( \Pi_3 = \)

\( \text{lattice of set partitions} \)

\( x_i = x_j \iff i, j \text{ in same block} \)
**Def’n:** The order complex of a finite poset $P$ is the simplicial complex $\Delta(P)$ whose $i$-dimensional faces are the $(i+1)$-chains in $P$.

**Example:**

- Let $\overline{P} = P \setminus \{0, 1\}$ e.g. for $\overline{\Pi}_n$

**Convention:** When we speak of topological properties (homology, etc.) of poset $P$, we mean $\Delta(P)$ or $\Delta(\overline{P})$.

**Poset rank:** = # steps from bottom
Goresky-MacPherson Formula

\[ \tilde{H}^i(M_A) = \bigoplus_{x \in \Lambda} \tilde{H}^{\text{codim}(x) - 2 - i}(\delta, x) \]

(subspace arrangement complement as groups, intersection lattice)

Plan: Apply to braid arrangement using upcoming \( S_n \)-equivariant version due to Sundaram-Welker, yielding Whitney homology. (See also Blagojević, Lück, Ziegler for more general versions)
$S_n$-Representations on Chains (i.e. on Faces) and on Homology

- $S_n$ action on set partitions is order-preserving and rank-preserving.

(Recall $P$ is graded if for each $u \leq v$ all saturated chains $u \rightarrow v$ have same length)

- Hence, induces $S_n$-action on $\{\text{chains } u_1 < u_2 < \ldots < u_j\}$

$\{\text{faces of } \Delta(\uparrow^n)\}$
\( S_n \)-action on chains commutes with simplicial boundary map

\[
d(u_0 < \ldots < u_r) = \sum_{i=0}^r (u_0 < \ldots < \hat{u}_i < \ldots < u_r)
\]

- Thus, \( S_n \)-action on \( i \)-faces (\( i \)th chain gp) induces rep'n on \( i \)th homology

- But homology of \( \overline{P}_n \) is concentrated in top degree due to EL-shellability of \( \overline{P}_n \)

\( \text{(since shellable} \Rightarrow \text{homotopy equivalent to wedge of spheres) } \)
**G-Equivariant Enrichment of\nGoresky-MacPherson Formula**

**Thm (Sundaram-Welker):** Let $A$ be a $G$-arrangement of $C$-linear subspaces in $C^n$ for $G$ a finite subgroup of $GL_n(C)$. Then

$$\tilde{H}^i(M_A) \cong \bigoplus_G \text{Ind}_{\text{Stab}(x)}^G \tilde{H}^i(\mathcal{O}_{x_0}^{(x)}, \text{codim}(x))$$

(in our case) = "WH_i(L_{A^n})"

\[ \cong \mathbb{P}^n \]

**Note:** there are numerous variations, e.g. allowing us also to handle config. spaces in $\mathbb{R}^{2d+1}$.
Whitney Homology (for Graded Posets)

\[ \text{WH}_i(P) := \text{"i-th Whitney homology of } P \text{"} \]
\[ = \bigoplus_{\substack{\mathcal{G}, \mathcal{U} \text{ s.t. } \mathcal{G} \prec \mathcal{U} \text{ has } \lambda \text{ non-nil blocks} \text{ and } \mathcal{U} \text{ is maximal}}} \tilde{H}_{i-2}(\mathcal{G}, \mathcal{U}) = \bigoplus_{\text{has } \lambda \text{ non-nil blocks}} \text{WH}_i(P) \]

\[ \text{WH}_2(P) := \bigoplus_{u \in P} \tilde{H}_{2\text{top}}(\mathcal{G}, u) \quad \text{type}(u) = \lambda \]

\( \lambda = (3,1,1) = 11s \text{ of block sizes} \)

\[ 123|4|5 \quad \rightarrow \quad 423|1|15 \quad \leftrightarrow \quad i = 2 \]

\[ 12|3|4|5 \quad 13|2|4|5 \quad 23|1|4|5 \quad \ldots \]

Aside: \( \text{WH}_i(P) \cong \bigoplus_{\lambda} \beta_{i, \lambda}(P) \oplus \delta_{i-1, \lambda}(P) \)
Thm (H-Reiner): Let $M_n^d$ = config. space of $n$ distinct pts in $\mathbb{R}^d$. Then $H^i(M_n^d)$ stabilizes sharply at $3i+1$.

More generally, $H^i(M_n^{2d})$ stabilizes sharply for $n \geq 3 \frac{i}{2d-1} + 1$ and $H^i(M_n^{2d+1})$ stabilizes sharply for $n \geq 3 \frac{i}{2d}$.

Idea: Determine stability of $\hat{\omega}_i \oplus \hat{\text{Lie}}_i$.

Thm (H-Reiner): $\langle H^i(M_n^d), S^{n-1\nu,\nu} \rangle$ vanishes for $|\nu| \leq 2i$ and becomes constant for $n \geq n_0 := \begin{cases} |\nu|+1 & \text{for } d \text{ odd} \\ |\nu|+i+1 & \text{for } d \text{ even} \end{cases}$
Proof Techniques & Results We'll Use

**Thm (Hanlon-Stanley):**  \( \prod_n \simeq \text{sgn} \otimes (\sum_i \xi_i^{s_i}) \)

**Thm (Joyal):**  \( \text{lie}_n \simeq \xi_i \)

**Cor:**  \( \prod_n \simeq \text{lie}_n \otimes \text{sgn} \)

**Thm (Krasikiewicz & Weyman):**

\[
\text{lie}_n \simeq \bigoplus \mathcal{S}^2(\tau)
\]

\[
\tau \text{ surj. w/ } |\tau| \equiv 1 \pmod{n}
\]

**Thm (Sundaram):**

\[
\text{ch}(WH_n) = \prod h_{m_j}^{[\pi_j]} \prod e_{m_j}^{[\pi_j]} \\
\text{\quad \quad \quad j odd} \quad \quad \quad j \text{ even}
\]

\[
=(h_{m_1}) \left( \prod h_{m_j}^{[\pi_j]} \right) \left( \prod e_{m_j}^{[\pi_j]} \right) \\
\text{\quad \quad \quad j odd} \quad \quad \quad j \geq 1 \quad \quad \quad j \text{ even}
\]
Thm (Sundaram):

\[ \text{ch}(WH_2) = \prod_j h_{m_j}^{\text{add}} \prod_j e_{m_j}^{\text{even}} \]

\[ = \left( h_{m_1} \right) \left( \prod_j h_{m_j} \right) \left( \prod_j e_{m_j} \right) \]

\[ \text{ch(triv}_m)= \]  

\[ \hat{W}_{2i} \] has degree \( \leq 2i \) by \( \ast \)

where \( \text{ch} = " \text{Frobenius characteristic" isom.} \)

\[ \text{ch}(f) = \sum_{\pi} f(\pi) \frac{P_\pi}{\xi_\pi} \] from \( S_n \)

class functions to ring of symmetric fn's

\[ h_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} \ldots x_{i_n} = \text{ch (trivial rep'n)} \]

\[ e_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_n} = \text{ch (sgn rep'n)} \]

Obs: \( \Pi_n \) has 1st row upper bd \( n-1 \) for \( n > 2 \) & \( e_m [\Pi_2] = e_m [h_2] \) has 1st row upper bd \( m+1 \)
* Key Fact for Stability: \( u \in \Omega_n \) of rank \( i \) has at most \( 2i \) letters in nontrivial blocks

**Significance**: Gives upper bound of \( 2i \) on \( |\lambda| \), where sharp stability bound is \( \max \{ |\lambda| + 2, 3 \} \)

\[
\begin{align*}
12|34|56|78 & \quad \text{max \# letters in nontriv. blocks} \\
\lambda &= (2,2,2,2) \\
12|34|56|7|8 & \quad 2\text{-rank} = 2i \\
\lambda &= (2,2,2,1,1) \\
12|34|5|6|7|8 & \quad \lambda = (3,1,1,1,1,1) \\
\end{align*}
\]
Wiltshire–Gordon Conjectures & Related Results

**Defn (Wiltshire–Gordon):**

\[ V_n^k = \bigoplus \text{WH}_\lambda (\Pi_n) \text{ strips away } \ell(\lambda) = n-k \text{ tensors w/trivial rep'n} \]

\( \lambda \) has no parts of size 1

**Thm (H-Reiner):**

\[ \text{Ind} (\text{Res} (V_n^k)) \oplus V_{n-1}^k \cong \text{Res} (V_{n+1}^k) \]

(conjectured by Wiltshire–Gordon)

*e.g.* \( n+1 = 5 \) \( k = 1 \) \( k = 3 \) dimension formula:

\[ 4 \cdot \left( \binom{4}{2} + (3-1)! \right) = \left( \binom{5}{3} \right) \cdot (3-1)! (2-1)! = 20 \]
**Key Question:** Decompose $V_n^k$ into irreducible reps, since this would exactly give the $S^3$ irrep's yielding $S^3 \otimes \text{triv}_{n-12}$ reps comprising $k$-th cohomology for config. space of $n$ distinct, labeled pts in $\mathbb{R}^2$.

**Progress (Next Theorem):** Answer instead for $\bigoplus_{k} V_n^k$.

**Open Qu:** Analogous results for $\mathbb{R}^d$ for $d>2$?
\textbf{Thm (H-Reiner)}:
\[ V_n = \text{ch} \left( \bigoplus V_n^k \right) \cong \bigoplus S^\lambda(T) \]

\[ T \text{ is "Whitney generating" SYT} \]

where \( T \) is \textbf{Whitney generating if either}

1. \( T = \emptyset \) or \[ \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \] or \[ \begin{array}{c}
1 \\
2 \\
3
\end{array} \]

or

2. \( T \mid \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \)

is one of the four shapes:

\[ T_1 = \begin{array}{c}
1 \\
3 \\
4
\end{array} \]
\[ T_2 = \begin{array}{c}
1 \\
2 \\
3 \\
4
\end{array} \]
\[ T_3 = \begin{array}{c}
1 \\
2 \\
3 \\
4 
\end{array} \quad T_4 = \begin{array}{c}
1 \\
2 \\
3 \\
4 
\end{array} \]

with the following further restrictions:

(a) If \( T_3 \), then the first ascent \( k \) with \( k \geq 4 \) is odd

(b) If \( T_4 \), then the first ascent \( k \) with \( k \geq 4 \) is even

**ascent**: \( i \) such that \( i + 1 \) in weakly higher row

**Idea**: Both sides satisfy same recurrence: categorified

\[ d_n = nd_{n-1} + 1 \]

\[ \hat{\omega H}_n = \hat{\omega H}_{n-1} \uparrow S_n + (-1)^n \nabla_n \]

for \( \nabla_n = \chi^{(3,1^n^3)} - \chi^{(2,3,1^n^2)} \) for \( n \geq 4 \)
Motivations from Number Theory for Repin Stability for PConf(\(\mathbb{R}^d\))

- Church-Ellenberg-Farb
- Matchett-Wood-Vakil, and others:

\[
\langle H^i_c(\text{PConf}_n(C)), V \rangle_{S_n} = \lim_{g} H^i_{et}(\text{Conf}_{n; g}) \quad \text{coefs twisted by } V
\]

yielding various counting formulas over finite field via "Grothendieck-Lefschetz formula" and counting fixed pts of Frobenius map

\[\lim_{n \to \infty} (\# D-free degree n polys) = 8^n - 8^{n-1}\]

Remarks: Applications to number theory focus on \(M = \mathbb{R}^2\) case

- We improve error bounds in these limits
**Thm:** $\langle \beta_S(TT_n), \text{triv} \rangle$ is constant for $n \geq 2\max(S) - (\frac{|S| - 1}{2})$.

**Note:** This follows from partitioning of $\Delta(TT_n)/S_n$ giving combinatorial interpretation for $\langle \beta_S(TT_n), \text{triv} \rangle$ (i.e. from 2003 result of H.), our point of entry to this topic.

**Conjecture (H- Reiner):** For fixed $S \subseteq \{1, 2, \ldots, n-2\}$ with $i = \max(S)$, the rank-selected homology $\beta_S(TT_n)$ stabilizes sharply at $n = 4i - |S| + 1$. 
**Thm (Sundaram):** \( S_j \)-rep’in on top homology of \( \Pi_j \)

\[
ch(WH_2) = \prod_j h_{m_j}^{\text{add}} \Pi_j \prod_j e_{m_j}^{\text{even}} \Pi_j
\]

\[
= (h_{m_j}) \left( \prod_j h_{m_j}^{\text{add}} \Pi_j \right) \left( \prod_j e_{m_j}^{\text{even}} \Pi_j \right)
\]

\[
ch(\text{trivm}_i) = \begin{cases} 
\text{"Wh}_i \text{" has degree } \leq 2i \text{ by } \\
\text{where } ch = "\text{Frobenius characteristic}" \text{ isom.}
\end{cases}
\]

\[
ch(f) = \sum_{\lambda} f(\lambda) \frac{P_\lambda}{\zeta_\lambda} \text{ from } S_n
\]

class functions to ring of symmetric fn's

\[
h_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} \ldots x_{i_n} = ch(\text{trivial rep'n})
\]

\[
e_n := \sum_{1 \leq i_1 < i_2 < \ldots} x_{i_1} x_{i_2} x_{i_3} \ldots x_{i_n} = ch(\text{sgn rep'n})
\]

**Obs:** \( \Pi_n \) has 1st row upper bd \( n-1 \) for \( n > 2 \) & \( e_m [\Pi_2] = e_m [h_2] \) has 1st row upper bd \( m+1 \)
Key Properties of Symmetric Functions

- \( S^\lambda \stackrel{ch}{\rightarrow} \) schur fn \( s_\lambda = \sum x^T \)

"Frobenius charact." \( T \) SSYT \( \lambda \) shape \( \lambda \)

\[
{\begin{array}{cccc}
1 & 1 & 2 & 2 \\
3 & 4 & & \\
\end{array}} \]

\[
T = \begin{pmatrix}
1 & 2 & 3 & x_1 x_2 x_3 x_4 \\
3 & 4 & & \\
\end{pmatrix}
\]

\( \Rightarrow \) \( s_\lambda \) includes monomial divisible by \( x_1^3 \) but not \( x_1^4 \).

- wreath \( \rightarrow \) plethysm of product symmetric functions of rep's

\( \Rightarrow \) \( f \) includes \( x_1^a \) \( g \) includes \( x_1^b \) then \( f \circ g \) includes \( x_1^{a+b} \) while \( f(g) \) cannot include \( x_1^{\deg f + \deg g} \).
Translating "Polynomial Characters" into Symmetric Fns (to get Improved Power Saving Bounds)

- Any polynomial \( P(x_1, x_2, x_3, \ldots) \) gives a class fn for \( S_n \) by letting \( x_i = \# i \)-cycles in conjugacy class.
- The elements \( (\lambda) = (m_1)(m_2)(\ldots) \) where \( \lambda \) has \( m_i \) parts of size \( i \) form a basis for \( \mathbb{Q}[x_1, x_2, x_3, \ldots] \).

Prop (H-Reiner): \( \text{ch}(x_P) = \sum \frac{P_\lambda}{z_\lambda} h_{n-|\lambda|} \) for \( n \gg 1 \)
for \( P = (\lambda) = (m_1)(m_2)(\ldots) \).
Combining with Earlier Results...

- guarantees for all $p \in \mathbb{Q}[x_1, x_2, \ldots]$
  
  $x_p = M \left( \sum_m c_m x^m \right)$ s.t. $1M1 \leq \text{deg}(p) \forall M$

- analyze $\langle x_p, H^i(M^{2d}) \rangle$ via:

\textbf{Thm (H-Reiner)}: $\langle H^i(M_n^{2d}), S^{(n-1\nu_1, \nu)} \rangle$

vanishes for $1\nu_1 \leq 2i$ and becomes

constant for $n \geq n_0 := \begin{cases} 1\nu_1 + i & \text{for } d \text{ odd} \\ 1\nu_1 + i + 1 \text{ for } d \text{ even} \end{cases}$

\textbf{Upshot}: $\langle x_p, H^i(\text{PConf}(C)) \rangle$ is constant for $n \geq \max \{ 2 \deg(p), \deg(p) + i + 1 \}$.